3D-2D analysis for the optimal elastic compliance problem

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Abstract

We study the asymptotic behaviour of the variational problems which consist in minimizing the compliance of a prescribed amount of elastic material under a given load, when the height of the design region becomes very small. We determine the limit problem, and we provide necessary and sufficient optimality conditions.

Résumé

French title.

Version française abrégée

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1. Introduction and setting of the problem

The aim of this note is to announce some results concerning the asymptotic behaviour of the $3D$ optimal elastic compliance problem, when the thickness of the design region tends to zero and the volume fraction remains unchanged. The complete proofs will appear in a forthcoming paper. We refer to [3,6] for some recent related works, and to the books [1,4,5] for a background on the modelling of thin plates.
Hereafter, we give a precise description of the variational problems we want to study (for the sake of clearness, we first state them on the varying design region of infinitesimal height, and then we provide their rescaled formulation on a fixed domain). The limit problem is depicted in Section 2. Results are collected in Section 3.
Let $Q := \overline{D} \times I$, where D is an open bounded connected subset of $\mathbb{R}^2$, and $I := [-1/2, 1/2]$. Let $\delta > 0$ be a varying infinitesimal parameter. We set $Q_\delta := \overline{D} \times \delta I = \overline{D} \times [-\delta/2, \delta, 2]$. We assume with no loss of generality that $|Q| = 1$, and we denote the spatial variable in $\mathbb{R}^3$ by $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$.

1.1. *The sequence of variational problems on the thinning domains Q_δ*

Let be given a system of loads $F = (F_1, F_2, F_3) \in H^{-1}(Q; \mathbb{R}^3)$, and an integrand $j : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$, which satisfy the following assumptions:

- F is "balanced", meaning that $\langle F, U\rangle_{\mathbb{R}^3} = 0$ whenever $e(U) := \nabla U + (\nabla U)^T = 0$
- j is convex, 2-homogeneous, and coercive.

The "resistance" offered to the load F by an elastic material having j as a stored energy density, when it is placed in a given subset Ω of the design region Q, is given by the opposite of the so-called *elastic compliance*. It can be expressed equivalently by one of the two following primal or dual formulations:

$$\begin{aligned}\mathcal{C}(\Omega, j, F) &:= \sup\Big\{ \langle F, U\rangle_{\mathbb{R}^3} - \int_\Omega j(e(U))\,dx \ : \ U \in \mathcal{C}^\infty(\mathbb{R}^3; \mathbb{R}^3) \Big\} \\ &= \inf\Big\{ \int_Q j^*(\Sigma)\,dx \ : \ \Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym}) \ , \ \Sigma = 0 \text{ on } Q \setminus \Omega \ , \ -\operatorname{div}(\Sigma) = F \Big\}\end{aligned} \tag{1}$$

(where j^* is the Fenchel conjugate of j, and $\mathbb{R}^{3\times 3}_{\rm sym}$ denotes the space of 3×3 symmetric real matrices).
Thus, if one is interested in finding the most resistant material configuration for a given volume fraction $\tau \in (0, 1)$, one has to solve the following infimum problem:

$$\inf\Big\{ \mathcal{C}(\Omega, j, F) \ : \ \Omega \subseteq Q \ , \ |\Omega| = \tau \Big\} \ . \tag{2}$$

Our goal is to study the asymptotic behaviour of such problem in the limit when the unit height of the design region is turned into an infinitesimal parameter δ, and under the assumption that the volume fraction τ remains fixed. Clearly, in order to obtain a meaningful limit problem, we need to scale also the system of loads. The properly scaled version of problem (2) on Q_δ reads

$$\mathcal{I}^\delta(\tau) := \inf\Big\{ \mathcal{C}(\Omega, j, \sqrt{\delta}\, F^\delta) \ : \ \Omega \subseteq Q_\delta \ , \ |\Omega| = \tau\delta \Big\} \ , \tag{3}$$

where $F^\delta \in H^{-1}(Q_\delta; \mathbb{R}^3)$ is defined by $F^\delta(x) := \big(\delta^{-1}F_1(x', \delta^{-1}x_3), \delta^{-1}F_2(x', \delta^{-1}x_3), F_3(x', \delta^{-1}x_3)\big)$.
It is convenient to remark that the volume constraint on the admissible sets in (3) can be eliminated by enclosing in the cost a volume penalization through a Lagrange multiplier. For a fixed $k \in \mathbb{R}^+$, we set

$$\phi^\delta(k) := \inf\Big\{ \mathcal{C}(\Omega, j, \sqrt{\delta}\, F^\delta) + \frac{k}{\delta}|\Omega| \ : \ \Omega \subseteq Q_\delta \Big\} \ . \tag{4}$$

Actually, if one is able to determine the asymptotics of $\phi^\delta(k)$, it is then easy to deduce the asymptotics of $\mathcal{I}^\delta(\tau)$ (see Corollary 3.2 (i)).

1.2. *The sequence of variational problems restated on the fixed domain Q*

In order to deal with variational problems formulated on the fixed design region Q, we need some change of variables. For every set $\Omega \subseteq Q_\delta$ admissible in (4), and for every pair (U, Σ) admissible respectively for the primal and dual formulation of $\mathcal{C}(\Omega, j, \sqrt{\delta}\, F^\delta)$, we set

$$\omega = \Big\{ \big(x', \delta^{-1} x_3\big) \ : \ (x', x_3) \in \Omega \Big\} \ , \tag{5}$$

$$U(x) = \Big(u_1\big(x', \delta^{-1}x_3\big),\ u_2\big(x', \delta^{-1}x_3\big),\ \delta^{-1} u_3\big(x', \delta^{-1}x_3\big) \Big) \ , \tag{6}$$

$$\Sigma(x) = \begin{bmatrix} \delta^{-1/2}\sigma_{\alpha\beta}(x', \delta^{-1}x_3) & \delta^{1/2}\sigma_{\alpha 3}(x', \delta^{-1}x_3) \\ \delta^{1/2}\sigma_{\alpha 3}(x', \delta^{-1}x_3) & \delta^{3/2}\sigma_{33}(x', \delta^{-1}x_3) \end{bmatrix} \ . \tag{7}$$

(here and in the sequel the indices α, β run from 1 to 2).
In terms of these new variables, one can check that there holds

$$\phi^\delta(k) = \inf \Big\{ \mathcal{C}^\delta(\omega) + k|\omega| \ : \ \omega \subset Q \Big\} \ , \tag{8}$$

being

$$\begin{aligned} \mathcal{C}^\delta(\omega) &:= \sup \Big\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\, dx \ : \ u \in \mathcal{C}^\infty(\mathbb{R}^3; \mathbb{R}^3) \Big\} \\ &= \inf \Big\{ \int_Q j^*(\Pi^\delta(\sigma))\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym}) \ , \ \sigma = 0 \text{ on } Q \setminus \omega \ , \ -\mathrm{div}(\sigma) = F \Big\} \ ; \end{aligned} \tag{9}$$

here the operators $e^\delta(u)$ and $\Pi^\delta(\sigma)$ are defined by

$$e^\delta(u) := \begin{bmatrix} e_{\alpha\beta}(u) & \delta^{-1}e_{\alpha 3}(u) \\ \delta^{-1}e_{\alpha 3}(u) & \delta^{-2}e_{33}(u) \end{bmatrix} \qquad \text{and} \qquad \Pi^\delta \sigma^\delta := \begin{bmatrix} \sigma_{\alpha\beta} & \delta\sigma_{\alpha 3} \\ \delta\sigma_{\alpha 3} & \delta^2\sigma_{33} \end{bmatrix} \ . \tag{10}$$

2. The limit optimization problem

In this section we state the limit problem of $\phi^\delta(k)$ in all its equivalent formulations. To this aim we need to introduce some notation.
We denote by $H^1_{KL}(Q; \mathbb{R}^3)$ the space of *Kirchoff-Love displacements*:

$$H^1_{KL}(Q; \mathbb{R}^3) := \Big\{ u \in H^1(Q; \mathbb{R}^3) \text{ such that } e_{i3}(u) = 0 \text{ for } i = 1, 2, 3 \Big\} \ ;$$

recall that any $u \in H^1_{KL}(Q; \mathbb{R}^3)$ may be rewritten as follows, for some $v_\alpha \in H^1(D)$ and $v_3 \in H^2(D)$:

$$u_\alpha(x) = v_\alpha(x') - \frac{\partial v_3}{\partial x_\alpha}(x') x_3 \ , \qquad u_3(x) = v_3(x') \ . \tag{11}$$

We denote by $\overline{j} : \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}$ the integrand obtained starting from j through the formula

$$\overline{j}(z) := \inf \Big\{ j\big(z + \sum_{i=1}^{3} \xi_i (e_i \otimes e_3)^*\big) \ : \ \xi_i \in \mathbb{R} \Big\} \ ; \tag{12}$$

in turn, starting from $\overline{j}$, we define $W_k : \mathbb{R}^{2\times 2}_{\text{sym}} \times \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}$ as

$$W_k(z, \xi) := \int_{-1/2}^{1/2} \big[\overline{j}(z - x_3 \xi) - k\big]_+ \, dx_3 \qquad (\ [\ \cdot\ ]_+ \text{ indicates the positive part}) \ . \tag{13}$$

For any given real measure ν on Q, we denote by $[\nu]$ the real measure on $\overline{D}$ defined by the identity $\langle [\nu], \varphi \rangle_{\mathbb{R}^2} := \langle \nu, \varphi \rangle_{\mathbb{R}^3}$ holding for all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R})$; then we set $\overline{F} = (\overline{F}_1, \overline{F}_2, \overline{F}_3)$, where:

$$\overline{F}_\alpha := [F_\alpha] \qquad \text{and} \qquad \overline{F}_3 := \big[F_3 + x_3 \sum_{\alpha=1}^{2} \frac{\partial F_\alpha}{\partial x_\alpha}\big] \ . \tag{14}$$

We are now ready to formulate the limit problem $\phi(k)$ of $\phi^\delta(k)$ as $\delta \to 0$. It can be written equivalently as a $3D$ variational problem in each of the following three variables:

$$\theta \in L^\infty(Q; [0,1]) \ , \qquad u \in H^1_{KL}(Q; \mathbb{R}^3) \ , \qquad \sigma \in L^2(Q; \mathbb{R}^{2\times 2}_{\text{sym}}) \ .$$

We give below the different definitions of $\phi(k)$ in terms of (θ, u, σ); they will be consistent with each other thanks to the convergence result stated in Section 3. The expressions of $\phi(k)$ in terms of u and σ are further complemented with their $2D$ counterpart, which is a variational problem respectively in the variables

$$v = (v_\alpha, v_3) \in (H^1(D))^2 \times H^2(D) \ , \qquad (\lambda, \eta) \in L^2(D; \mathbb{R}^{2\times 2}_{\text{sym}}) \ ;$$

the variable v is related to u by (11), whereas (λ, η) are related to σ by the identities $[\sigma] = \lambda$, $[-x_3 \sigma] = \eta$.

I. The $3D$ limit problem in θ.

$$\begin{aligned}
\phi(k) &:= \inf \Big\{ \mathcal{C}(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q; [0,1]) \Big\} \ , \text{ where} \\
\mathcal{C}(\theta) &:= \sup \Big\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{\alpha\beta}(u))\, \theta \, dx \ : \ u \in H^1_{KL}(Q; \mathbb{R}^3) \Big\} \\
&= \inf \Big\{ \int_Q \theta^{-1} j^*(\sigma) \, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\text{sym}}) \ , \ \sigma_{3i} = 0 \ \forall i \ , \ \operatorname{div} \sigma + F \in \big(H^1_{KL}(Q; \mathbb{R}^3)\big)^\perp \Big\} \ .
\end{aligned} \tag{15}$$

II. The $3D$ limit problem in u (and its $2D$ counterpart in v).

$$\begin{aligned}
\phi(k) &:= \sup \Big\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j}(e_{\alpha\beta}(u)) - k\big]_+ \, dx \ : \ u \in H^1_{KL}(Q; \mathbb{R}^3) \Big\} \\
&= \sup \Big\{ \langle \overline{F}, v \rangle_{\mathbb{R}^2} - \int_D W_k(e(v^1, v^2), \nabla^2 v_3) \, dx' \ : \ v_1, v_2 \in H^1(D) \ , \ v_3 \in H^2(D) \Big\} \ .
\end{aligned} \tag{16}$$

III. The $3D$ limit problem in σ (and its $2D$ counterpart in (λ, η)).

$$\begin{aligned}\phi(k) &:= \inf\Big\{ \int_Q [\overline{j}-k]^*_+(\sigma)\,dx \ :\ \sigma\in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})\ ,\ -{\rm div}[\sigma]=(\overline{F}_1,\overline{F}_2)\ ,\ -{\rm div}^2[x_3\sigma]=\overline{F}_3\Big\} \\ &= \inf\Big\{ \int_D W^*_k(\lambda,\eta)\,dx' \ :\ \lambda,\eta\in L^2(D;\mathbb{R}^{2\times 2}_{\rm sym})\ ,\ -{\rm div}\,\lambda=(\overline{F}_1,\overline{F}_2)\ ,\ {\rm div}^2\,\eta=\overline{F}_3\Big\}\ .\end{aligned} \tag{17}$$

Notice that, in contrast with ϕ^δ, the function ϕ is convex in k (as its expression in (16) shows); in particular, ϕ is differentiable except possibly at countably many values of k.

3. The results

Our main results are the convergence and the optimality condition statements given respectively in Theorems 3.1 and 3.3 below.
We adopt the notation χ_ω for the characteristic function of a set $\omega\subseteq Q$, and $E_0:\mathbb{R}^{2\times2}_{\rm sym}\to\mathbb{R}^{3\times3}_{\rm sym}$ for the linear operator defined by $E_0\sigma := \begin{bmatrix}\sigma_{11} & \sigma_{12} & 0\\ \sigma_{21} & \sigma_{22} & 0\\ 0 & 0 & 0\end{bmatrix}$.

Theorem 3.1 (**convergence**) *For a fixed $k\in\mathbb{R}$, let $\phi^\delta(k)$ be defined by* (8)*, and let $\phi(k)$ be defined by any of the formulae* (15), (16), (17)*. Let $\omega^\delta\subset Q$ be a minimizing sequence for* (8) *(that is, $\mathcal{C}^\delta(\omega^\delta)+k|\omega^\delta|-\phi^\delta(k)=o(1)$), and for each δ let $\sigma^\delta\in L^2(Q;\mathbb{R}^{2\times2})$ be optimal for the dual formulation of $\mathcal{C}^\delta(\omega^\delta)$ (cf. the infimum problem in the second line of* (9)*). Then, up to subsequences, there holds*

(i) $\lim_{\delta\to0}\phi^\delta(k)=\phi(k)$;

(ii) $\lim_{\delta\to0}\chi_{\omega^\delta}=\overline{\theta}$ *weakly * in* $L^\infty(Q;[0,1])$, *where $\overline{\theta}$ is optimal for problem* (15);

(iii) $\lim_{\delta\to0}\Pi^\delta\sigma^\delta=E_0\overline{\sigma}$ *weakly in* $L^2(Q;\mathbb{R}^{3\times3})$, *where $\overline{\sigma}$ is optimal for problem* (17).

The proof of Theorem 3.1 is based on the use of so-called ficticious materials: the basic idea is to show that the sequence $\phi^\delta(k)$ behaves asymptotically as its ficticious counterpart $\tilde{\phi}^\delta(k):=\inf\big\{\mathcal{C}^\delta(\theta)+k\int_Q\theta\,dx\ :\ \theta\in L^\infty(Q;[0,1])\big\}$, where the extended compliance $\mathcal{C}^\delta(\theta)$ is obtained simply by replacing the characteristic function χ_ω by the varying density θ in the integral appearing in the first line of (9). Once established the asymptotic equivalence between $\phi^\delta(k)$ and $\tilde{\phi}^\delta(k)$, the limiting behaviour of the latter can be studied by means of rather standard tools in the Calculus of Variations (such as Γ-convergence and duality methods). In the ficticious framework, it is also possible to establish the weak H^1 convergence of the optimal strain displacements u^δ for $\tilde{\phi}^\delta(k)$ to an optimal $\overline{u}$ for problem (16).

As a consequence of Theorem 3.1, we may firstly determine the asymptotics as $\delta\to0$ of the volume constrained problems given by (3); and secondly, when the volume fraction τ becomes infinitesimal, we may recover the same limit problem obtained through a different approach in [2].

Corollary 3.2 *For a fixed $\tau\in(0,1)$, let $\mathcal{I}^\delta(\tau)$ be defined by* (3)*. Then there holds*

(i) $\lim_{\delta\to0}\mathcal{I}^\delta(\tau)=\mathcal{I}(\tau):=\sup_{k\in\mathbb{R}^+}\Big\{\Phi(k)-k\tau\Big\}$

(ii) $\lim_{\tau\to0}\tau\mathcal{I}(\tau)=\dfrac{{\mathcal{S}_0}^2}{2}$ *where* $\mathcal{S}_0:=\sup\Big\{\langle\overline{F},v\rangle_{\mathbb{R}^2}\ :\ v\in\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^3)\ ,\ \overline{j}\big(e(v_2,v_2)\pm\frac{1}{2}\nabla^2v_3\big)\le\frac{1}{2}\Big\}$.

We now turn attention to the optimality conditions. For brevity we state them just in terms of the $3D$ variables (θ, u, σ), but they may be reformulated also replacing u and σ respectively by the $2D$ variables v and (λ, η).

Theorem 3.3 **(optimality conditions)** *A triple* $(\overline{\theta}, \overline{u}, \overline{\sigma})$ *which is admissible in* (15)-(16)-(17) *is optimal for the corresponding problems if and only if it satisfies the conditions:*

$$\overline{\sigma} \in \partial\Big([\overline{j}(e_{\alpha\beta}(\overline{u})) - k]_+\Big) \qquad \text{and} \qquad \overline{\theta}\,\big(\overline{j}(e_{\alpha\beta}(\overline{u})) - k\big) = [\overline{j}(e_{\alpha\beta}(\overline{u})) - k]_+ \ .$$

In particular, on the set $\{x \in Q \ : \ \overline{j}(e_{\alpha\beta}(\overline{u})) < k\}$*, there holds* $\overline{\theta} = 0$ *and* $\overline{\sigma} = 0$ *, whereas on the set* $\{x \in Q \ : \ \overline{j}(e_{\alpha\beta}(\overline{u})) > k\}$*,* $\overline{\theta} = 1$ *and* $\overline{\sigma}(x', \cdot)$ *is affine.*

Corollary 3.4 *The optimization problem* (15) *admits at least a solution* $\overline{\theta}$ *which takes values into* $\{0, 1\}$*. Furthermore, if* ϕ *is differentiable at* k *(that is for all* k *except possibly countably many values),* $\overline{\theta}$ *is unique, the whole sequence* χ_{ω^δ} *in Theorem 3.1 converges to* $\overline{\theta}$*, and there holds* $\overline{\theta} = \chi_{\overline{\omega}}$*, where* $\overline{\omega} := \{x \in Q \ : \ \overline{j}(e_{\alpha\beta}(\overline{u})(x)) > k\}$*, being* $\overline{u}$ *is any solution to problem* (16).

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Notes

- add assumptions on the loads?
- clarify the definition domain for the parameters τ and k
- check all the scalings in section 1.2
- check signs in front of div^2
- add more items in the bibliography?