Optimal thin torsion rods and Cheeger sets

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Abstract

We carry out the asymptotic analysis of the following shape optimization problem: a given volume fraction of elastic material must be distributed in a cylindrical design region of infinitesimal cross section in order to maximize the resistance to a twisting load. We derive a limit rod model written in different equivalent formulations and for which we are able to give necessary and sufficient conditions characterizing optimal configurations. Eventually we show that, for a convex design region and for very small volume fractions, the optimal shape tends to concentrate section by section near the boundary of the Cheeger set of the design. These results were announced in [11].

1 Introduction

Let Q be a given design region in $\mathbb{R}^3$, and let G be a given load in $H^{-1}(Q;\mathbb{R}^3)$. When an isotropic elastic material occupies a region $\Omega \subset Q$, its *compliance* $\mathcal{C}(\Omega)$ is defined by

$$\mathcal{C}(\Omega) := \sup \Big\{ \langle G, u \rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\, dx \ : \ u \in H^1(Q;\mathbb{R}^3) \Big\} , \tag{1.1}$$

where, as usual in linear elasticity, $e(u)$ denotes the symmetric part of ∇u and the strain potential j, assumed to be isotropic, has the form $j(z) = (\lambda/2)(\mathrm{tr}(z))^2 + \eta |z|^2$ (see [15]). The Lamé coefficients satisfy the conditions $\eta > 0$ and $3\lambda + 2\eta > 0$, which ensure the strict convexity of j.
Clearly, in order that $\mathcal{C}(\Omega)$ remains finite, G must have support contained into $\overline{\Omega}$, and be a *balanced load*, meaning:

$$\langle G, u \rangle_{\mathbb{R}^3} = 0 \text{ whenever } e(u) = 0 .$$

Under this condition, an optimal displacement field $\overline{u}$ in problem (1.1) exists, and $\mathcal{C}(\Omega) = \frac{1}{2}\langle G, \overline{u} \rangle_{\mathbb{R}^3}$. This shows that the compliance is proportional to the work done by G in order to bring the structure to equilibrium and therefore finding the most robust configurations of a prescribed amount of material requires minimizing the shape functional $\mathcal{C}(\Omega)$ under a volume constraint on the admissible domains Ω:

$$\inf \Big\{ \mathcal{C}(\Omega) \ : \ \Omega \subseteq Q \ , \ |\Omega| = m \Big\} . \tag{1.2}$$

It is well known that this variational problem is in general ill-posed due to homogenization phenomena which prevent the existence of an optimal domain (see [2]), so that relaxed solutions must be searched under the form of densities with values in $[0, 1]$.

In some recent papers, we have focused our attention on the limiting behaviour of problem (1.2) when the design region Q is an "asymptotically thin" cylinder Q_δ of the form

$$Q_\delta = \overline{D} \times \delta I \qquad \text{or} \qquad Q_\delta = \delta\overline{D} \times I \ ,$$

where $\delta > 0$ is a small parameter, $I = [-1/2, 1/2]$ is a bounded interval, and $D \subset \mathbb{R}^2$ is an open bounded connected domain.
The case when $Q_\delta = D \times \delta I$ corresponds to perform a $3d$-$2d$ dimension reduction in problem (1.2) and to study the optimal design of less compliant thin plates (see [7, 8, 9, 10]).
The case when $Q_\delta = \delta D \times I$, which is quite far from being merely a technical variant of the previous one, involves a $3d$-$1d$ dimension reduction process: the matter is now the optimization of thin elastic rods. This is the object of the present paper, where we prove the results announced in [11].

If for convenience we enclose the volume constraint in the cost through a Lagrange multiplier $k \in \mathbb{R}$, the sequence of variational problems under study takes the form:

$$\phi^\delta(k) := \inf_{\Omega \subseteq \delta\overline{D} \times I} \left\{ \sup_u \left[\langle G^\delta, u \rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\, dx \right] + \frac{k}{\delta^2} |\Omega| \right\} . \tag{1.3}$$

Here G^δ is a suitable scaling of G, chosen so that in the limit process the infimum will remain finite. Moreover, since in this paper we focus our attention on rods in pure torsion regime, G^δ will be chosen so that only twist displacement fields will be involved in the limit as $\delta \to 0$.

The paper is organized as follows.

In Section 2 we set up all the preliminaries, concerning in particular twist displacement fields and the class of torsion loads under consideration (see Definition 2.1).

In Section 3, we determine the limit $\phi(k)$ of $\phi^\delta(k)$ as $\delta \to 0^+$, under the form of a convex, well-posed problem for material densities $\theta \in L^\infty(Q; [0, 1])$ (see Theorem 3.2). We point out that the dimension reduction process is performed without making any topological assumption on the set Ω occupied by the material. Therefore, it is not covered by the very extensive literature on $3d$-$1d$ analysis which we give up to quote (we limit ourselves to mention [21, 23, 24, 25, 27] and references therein). The proof is based on the comparison with the "fictitious counterpart" to (1.3) (see (3.8)). The main ingredients are some delicate compactness properties derived from variants of the Korn inequality, and a crucial bound for the relaxed functional of the compliance established in [10, Proposition 2.8].

In Section 4, we give reformulations of $\phi(k)$ as a variational problem for twist displacement fields, as well as a variational problem for stress tensors (see respectively Theorem 4.1 and Theorem 4.2). This allows to give explicit necessary and sufficient optimality conditions (see Theorem 4.5). In particular, by exploiting the optimality system, the question whether the density formulations of $\phi(k)$ admits a classical solution (*i.e.* a density with values in $\{0, 1\}$) can be rephrased in a very simple way. We believe this is an interesting open problem, see Remark 4.6.

Finally in Section 5, we enlighten the role of Cheeger sets in the limiting behaviour of $\phi(k)$ as $k \to +\infty$ (see Theorems 5.2 and 5.4). As explained within a different context in [2, Section 4.2.3] (see also [10, Section 6]), considering large values of k corresponds to considering a small "filling ratio" $|\Omega|/|Q|$. It turns out that, when the cross section D is convex, as $k \to +\infty$ the material tends to concentrate section by section near the boundary of the so-called *Cheeger set* of D. Such set is determined by solving a purely geometric problem which in the last years has captured the attention of many authors (see [3, 12, 13, 14, 16, 17, 18, 19]): in general, if D is an open connected

set in the plane, a Cheeger set of D is a minimizer, if it exists, for the quotient perimeter/area among all subsets of D having finite perimeter.
To the best of our knowledge, until now there was no rigorous statement and proof for this geometric characterization of optimal "light" torsion rods. Let us emphasize that such characterization is valid only in pure torsion. For more general loads, due to the interplay between the bending, twisting and stretching energies, we foresee a much more complicated rod model, which is beyond the scopes of this paper.

Let us finally point out that, to make the paper more readable, the proofs of technical or auxiliary lemmas have been postponed to the Appendix.

2 Preliminaries

2.1 Notation

Throughout the paper we adopt the following conventions.
We let the Greek indices α and β run from 1 to 2, the Latin indices i and j run from 1 to 3, and as usual we omit to indicate the sum over repeated indices.
We set $Q = \overline{D} \times I$, where $I = [-1/2, 1/2]$ and D is an open, bounded, connected subset of $\mathbb{R}^2$ with a Lipschitz boundary.
We write any $x \in \mathbb{R}^3$ as $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, and we choose the coordinate axes so that $\int_D x_\alpha \, dx' = 0$.
Derivation of functions depending only on x_3 will be denoted by a prime.
The characteristic function of a set A, which equals 1 in A and 0 outside, is denoted by $\mathbb{1}_A$.
Whenever we consider distributions or functions with a compact set as definition domain, we implicitly mean they are extended to zero outside. In particular, Sobolev maps on Q or D are intended as the restrictions to Q or D of Sobolev maps on $\mathbb{R}^3$ or $\mathbb{R}^2$: this definition agrees with the usual one thanks to the boundary regularity assumed on D.
When we add a subscript $_m$ to a functional space, we are considering the subspace of its elements which have zero integral mean.
For any $T \in \mathcal{D}'(\mathbb{R}^3)$, we denote by $[[T]] \in \mathcal{D}'(\mathbb{R})$ the $1d$-average distribution defined by the identity

$$\langle [[T]], \varphi \rangle_{\mathbb{R}} := \langle T, \varphi \rangle_{\mathbb{R}^3} \qquad \forall \varphi = \varphi(x_3) \in \mathcal{C}_0^\infty(\mathbb{R}) \ .$$

2.2 Displacement fields

As usual, by *rigid motion* we mean any displacement with null symmetric gradient, hence of the form $a + b \wedge x$, with a, $b \in \mathbb{R}^3$.
We call *Bernoulli-Navier field* any displacement in the space

$$BN(Q) := \Big\{ u \in H^1(Q; \mathbb{R}^3) \ : \ e_{ij}(u) = 0 \quad \forall (i,j) \neq (3,3) \Big\} \ .$$

It is easy to check that, up to subtracting a rigid motion, any $u \in BN(Q)$ admits the representation

$$u_\alpha(x) = \zeta_\alpha(x_3) \ , \ u_3(x) = \zeta_3(x_3) - x_\alpha \zeta_\alpha'(x_3) \quad \text{for some } (\zeta_\alpha, \zeta_3) \in (H^2_m(I))^2 \times H^1_m(I) \ . \tag{2.1}$$

Further, we introduce the following space of displacements

$$TW(Q) := \Big\{ v = (v_\alpha, v_3) \in H^1(Q; \mathbb{R}^2) \times L^2(I; H^1_m(D)) \ : \ e_{\alpha\beta}(v) = 0 \quad \forall \alpha, \beta \in \{1,2\} \Big\} \ ,$$

which is the direct sum of $BN(Q)$ and of *twist fields*, namely displacements of the form

$$(v_1, v_2) = c(x_3)(-x_2, x_1) \quad \text{for some } c \in H^1_m(I) \ , \quad v_3 \in L^2(I; H^1_m(D)) \ . \tag{2.2}$$

Notice that the third component v_3 of a field in $TW(Q)$ is not necessarily in $H^1(Q)$; nevertheless, using the representation (2.2), we see that

$$(e_{13}(v), e_{23}(v)) = \frac{1}{2} \left(c'(x_3)(-x_2, x_1) + \nabla_{x'} v_3 \right) \in L^2(Q; \mathbb{R}^2) \ . \tag{2.3}$$

2.3 Admissible loads

We now fix the type of exterior loads we consider in this paper.

Definition 2.1 *We say that $G \in H^{-1}(Q; \mathbb{R}^3)$ is an admissible torsion load if*

$$G = \operatorname{div} \Sigma \textit{ for some } \Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym}) \textit{ with } \Sigma_{33} = 0 \tag{2.4}$$

$$\{x \in Q \ : \ \operatorname{dist}(x, \operatorname{spt}(G)) < \delta\} \textit{ has vanishing Lebesgue measure as } \delta \to 0 \ . \tag{2.5}$$

Lemma 2.2 and Remark 2.3 concern respectively assumptions (2.4) and (2.5). Subsequently, we give some typical examples of admissible torsion loads.

Lemma 2.2 *The loads G which fulfill assumption* (2.4) *form a vector subspace of $H^{-1}(Q; \mathbb{R}^2) \times L^2(I; H^{-1}(D))$. Such loads do not act on rigid motions, nor on Bernoulli-Navier displacements, whereas their action on any $v \in TW(Q)$ represented as in* (2.2) *is given by*

$$\langle G, v \rangle_{\mathbb{R}^3} = \langle m_G, c \rangle_{\mathbb{R}} + \langle G_3, v_3 \rangle_{\mathbb{R}^3} \ , \tag{2.6}$$

where $m_G \in H^{-1}(I)$ denotes average momentum of (G_1, G_2), defined by

$$m_G := [[x_1 G_2 - x_2 G_1]] \ .$$

Proof. Assumption (2.4) means that there exists $\Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym})$ with $\Sigma_{33} = 0$ such that the following system is satisfied in $\mathcal{D}'(\mathbb{R}^3)$:

$$\begin{cases} \partial_i \Sigma_{1i} = G_1 \\ \partial_i \Sigma_{2i} = G_2 \\ \partial_\alpha \Sigma_{3\alpha} = G_3 \ , \end{cases} \tag{2.7}$$

or equivalently

$$\langle G, u \rangle_{\mathbb{R}^3} = -\langle \Sigma, e(u) \rangle_{\mathbb{R}^3} \qquad \forall\, u \in H^1(Q; \mathbb{R}^3) \ . \tag{2.8}$$

By (2.8), it is clear that G is balanced, namely it vanishes on rigid displacements. More generally, since $\Sigma_{33} = 0$, G vanishes on Bernoulli-Navier displacements.
On the other hand, the action of G on twist displacements is well-defined through the equality

$$\langle G, v \rangle_{\mathbb{R}^3} = -2 \langle \Sigma_{\alpha 3}, e_{\alpha 3}(v) \rangle_{\mathbb{R}^3} \quad \forall v \in TW(Q) \ , \tag{2.9}$$

where the right hand side makes sense as a scalar product in $L^2(Q; \mathbb{R}^2)$ thanks to (2.3). In particular, by taking $v = (0, 0, v_3)$, one can see that $G_3 \in L^2(I; H^{-1}(D))$. Finally, representing twist fields v as in (2.2), equality (2.9) can be rewritten under the form (2.6). □

Remark 2.3 Assumption (2.5) is needed to ensure that the load can be supported by a small amount of material. From a technical point of view, (2.5) enables us to apply Proposition 2.8 in [10]. This condition on the topological support of G is satisfied for instance when $\mathrm{spt}(G)$ is a 2-rectifiable set, and in particular in the standard case when G is applied at the boundary of Q.

Example 2.4 *(Horizontal load concentrated on the "top and bottom faces" $D \times \{-1/2, 1/2\}$)*
For $\rho \in BV(I)$ and $\psi \in H^1_0(D)$, consider the horizontal load

$$(G_1, G_2) = \rho'(x_3)(-\partial_2\psi(x'), \partial_1\psi(x')) \ , \ G_3 = 0 \ .$$

Assumption (2.4) is readily satisfied by taking

$$\Sigma_{\alpha\beta} = 0 \quad \text{ and } \quad (\Sigma_{13}, \Sigma_{23}) = \rho(x_3)(-\partial_2\psi(x'), \partial_1\psi(x')) \ .$$

Hence G is an admissible load provided (2.5) holds, which happens as soon as ρ is piecewise constant. In particular, the choice $\rho(x_3) = 1\!\!1_I(x_3)$ corresponds to applying a surface force on the top and bottom faces of the cylinder Q. If in addition D is a circular disk of radius R and we take $\psi(x') = \frac{R^2 - |x'|^2}{2}$, we obtain the classical boundary load in torsion problem, that is

$$(G_1, G_2) = (\delta_{1/2} - \delta_{-1/2})(x_3)(-x_2, x_1) \ ,$$

being δ_a the Dirac mass at $x = a$. In this case the average momentum of (G_1, G_2) is given by

$$m_G = \frac{\pi R^4}{2} \, (\delta_{1/2} - \delta_{-1/2})(x_3) \ .$$

Example 2.5 *(Horizontal load concentrated on the "lateral surface" $\partial D \times I$)*
Denote by $\tau_{\partial D}$ the unit tangent vector at ∂D. For any $\rho \in L^2_m(I)$, the following horizontal load supported on $\partial D \times I$ is admissible:

$$(G_1, G_2) = \rho(x_3)(-\partial_2 1\!\!1_D(x'), \partial_1 1\!\!1_D(x')) = \rho(x_3)\tau_{\partial D}(x')\mathcal{H}^1 \llcorner \partial D \ , \ G_3 = 0 \ .$$

In order to check assumption (2.4), we choose $\psi \in H^1_0(D)$ such that $\int_D \psi = |D|$, and we decompose G as $G' + G''$, being

$$(G'_1, G'_2) := \rho(x_3)(-\partial_2\psi(x'), \partial_1\psi(x')) \ , \ G'_3 = 0 \ ,$$

and $G'' := G - G'$. Since the class of loads satisfying (2.4) form a linear space, it is enough to show that system (2.7) is solvable separately for G' and G''.
For G', this is true as already shown in Example 2.4. Concerning G'', we may rewrite it as

$$(G''_1, G''_2) = \rho(x_3)(F_1(x'), F_2(x')) \ , \quad G''_3 = 0 \ ,$$

where $(F_1, F_2) := (-\partial_2(1\!\!1_D - \psi), \partial_1(1\!\!1_D - \psi))$. Since by construction (F_1, F_2) is a balanced load in $H^{-1}(D; \mathbb{R}^2)$, there exists a solution $\sigma \in L^2(D; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ to the equation $\mathrm{div}\,\sigma = (F_1, F_2)$. Then system (2.7) is satisfied by taking

$$\Sigma_{\alpha\beta} = \rho(x_3)\sigma_{\alpha\beta}(x') \quad \text{ and } \quad \Sigma_{\alpha 3} = 0 \ .$$

We notice that in this example the average momentum is absolutely continuous with respect to the Lebesgue measure, more precisely

$$m_G = -2|D|\rho(x_3) \ .$$

Example 2.6 *(Load concentrated on the whole boundary of Q)*
Let $h \in L^2_m(\partial D)$, and let $\psi \in H^1(D)$ be the solution of the two-dimensional Neumann problem

$$\begin{cases} \Delta\psi = 0 & \text{in } D\,, \\ \partial_\nu \psi = h & \text{on } \partial D\,. \end{cases}$$

The following load (which is supported on the whole boundary of Q and in particular is purely vertical on its lateral surface) is admissible:

$$(G_1, G_2) = (\delta_{-1/2} - \delta_{1/2})(x_3)\, \nabla_{x'}\psi(x')\ , \quad G_3 = -h\, \mathcal{H}^1 \llcorner \partial D\ .$$

Indeed, the system (2.7) is satisfied by taking

$$\Sigma_{\alpha\beta} = 0 \quad \text{and} \quad \Sigma_{\alpha 3} = 1\!\!1_Q(x)\partial_\alpha\psi(x')\ .$$

The average momentum of (G_1, G_2) is given by

$$m_G = \left(\int_D \nabla_{x'}\psi \cdot (-x_2, x_1)\, dx'\right)(\delta_{-1/2} - \delta_{1/2})(x_3)\ .$$

3 The small cross section limit

In this section, for a fixed $k \in \mathbb{R}$, we are going to establish the asymptotics of the sequence $\phi^\delta(k)$ in (1.3) as $\delta \to 0$. To this aim, it is convenient to reformulate (1.3) as a shape optimization problem on the fixed domain Q in place of the thin cylinder $Q_\delta = \delta\overline{D} \times I$. In this respect let us precise that, throughout the paper, the scaling of the load is chosen as follows:

$$G^\delta(x) := \left(\delta^{-1}G_\alpha(\delta^{-1}x', x_3), \delta^{-2}G_3(\delta^{-1}x', x_3)\right)\ .$$

Further, let us introduce the operator $e^\delta : H^1(Q;\mathbb{R}^3) \to L^2(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ defined by

$$e^\delta_{\alpha\beta}(u) := \delta^{-2}e_{\alpha\beta}(u)\ , \quad e^\delta_{\alpha 3}(u) := \delta^{-1}e_{\alpha 3}(u)\ , \qquad e^\delta_{33}(u) := e_{33}(u)\ ,$$

as it is usual in the literature on $3d$-$1d$ dimension reduction.

Lemma 3.1 *Problem* (1.3) *can be reformulated as*

$$\phi^\delta(k) = \inf\left\{\mathcal{C}^\delta(\omega) + k|\omega| \ : \ \omega \subseteq Q\right\}\ , \tag{3.1}$$

where

$$\mathcal{C}^\delta(\omega) := \sup\left\{\delta^{-1}\langle G, u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\, dx \ : \ u \in H^1(Q;\mathbb{R}^3)\right\}\ . \tag{3.2}$$

Proof. See Appendix.

Now, in order to write down the limit problem of the sequence $\phi^\delta(k)$ in (3.1), we need to introduce the *reduced potential* according to the formula

$$\bar{j}(y) := \inf_{\xi_{\alpha\beta}\in\mathbb{R}} j \begin{bmatrix} \xi_{11} & \xi_{12} & y_1 \\ \xi_{12} & \xi_{22} & y_2 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad \forall y \in \mathbb{R}^3\ . \tag{3.3}$$

Recalling that $j(z) = (\lambda/2)(\mathrm{tr}(z))^2 + \eta|z|^2$, some explicit computations give

$$\overline{j}(y) = 2\eta \sum_{\alpha} |y_\alpha|^2 + (Y/2)|y_3|^2 \ , \tag{3.4}$$

where $Y = \eta\frac{3\lambda+2\eta}{\lambda+\eta}$ is the Young modulus, written in terms of the Lamé coefficients λ, η. In particular, in the limit problem, we shall need to compute $\overline{j}$ just at vectors of the form $(y_1, y_2, 0)$, which gives simply $2\eta|y|^2$.

The behaviour of the optimal design problem (3.1) in the dimension reduction process is described by the following result.

Theorem 3.2 *Let $G \in H^{-1}(Q;\mathbb{R}^3)$ be an admissible torsion load according to Definition 2.1. For every fixed $k \in \mathbb{R}$, as $\delta \to 0$, the sequence $\phi^\delta(k)$ in* (3.1) *converges to the limit $\phi(k)$ defined by*

$$\phi(k) := \inf \left\{ \mathcal{C}^{lim}(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} \ , \tag{3.5}$$

where

$$\mathcal{C}^{lim}(\theta) := \sup \left\{ \langle G, v\rangle_{\mathbb{R}^3} - \int_Q \overline{j}\big(e_{13}(v), e_{23}(v), 0\big)\, \theta \, dx \ : \ v \in TW(Q) \right\} \tag{3.6}$$

$$= \sup \Big\{ \langle m_G, c\rangle_{\mathbb{R}} + \langle G_3, w\rangle_{\mathbb{R}^3} - \frac{\eta}{2} \int_Q \big| c'(x_3)(-x_2, x_1) + \nabla_{x'} w \big|^2 \, \theta \, dx \ : \ c \in H^1_m(I), w \in L^2(I; H^1_m(D)) \Big\}. \tag{3.7}$$

*Moreover, if $\omega^\delta \subseteq Q$ is a sequence of domains such that $\phi^\delta(k) = \mathcal{C}^\delta(\omega^\delta) + k|\omega^\delta| + o(1)$, then, up to subsequences, $1\!\!1_{\omega^\delta}$ converges weakly * in $L^\infty(Q;[0,1])$ to a solution $\overline{\theta}$ of problem* (3.5).

The remaining of Section 3 is entirely devoted to the proof of Theorem 3.2. It is based on the idea of considering the "fictitious counterpart" of problem (3.1), namely

$$\widetilde{\phi}^\delta(k) := \inf \left\{ \widetilde{\mathcal{C}}^\delta(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} \ , \tag{3.8}$$

where $\widetilde{\mathcal{C}}^\delta(\theta)$ denotes the natural extension of the compliance $\mathcal{C}^\delta(\omega)$ to $L^\infty(Q;[0,1])$:

$$\widetilde{\mathcal{C}}^\delta(\theta) := \sup \left\{ \delta^{-1} \langle G, u\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u))\, \theta \, dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\} \ . \tag{3.9}$$

For the sake of clearness, we divide the proof in three parts. In Part I we establish some delicate compactness properties which are preliminary to Part II, where we show that the sequence $\widetilde{\phi}^\delta(k)$ converges to the limit problem $\phi(k)$ given by (3.5). We conclude by showing in Part III that the sequences $\phi^\delta(k)$ and $\widetilde{\phi}^\delta(k)$ have the same asymptotics.

3.1 Part I: compactness

We start with a key lemma: it enlightens the role of condition (2.4) appearing in the definition of admissible torsion load.

Lemma 3.3 *Let $\theta \in L^\infty(Q;[0,1])$ with $\inf_Q \theta > 0$, and let $G \in H^{-1}(Q;\mathbb{R}^3)$ be an admissible torsion load, with $G = \operatorname{div}\Sigma$ as in (2.4). If $u^\delta \in \mathcal{C}^\infty(Q;\mathbb{R}^3)$ is a sequence such that*

$$\inf_\delta \left\{ \delta^{-1}\langle G, u^\delta\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u^\delta))\theta\, dx \right\} > -\infty \ , \tag{3.10}$$

then the sequence $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$. Moreover, if we pass to a subsequence such that $\lim_\delta e^\delta_{\alpha 3}(u^\delta) = \chi_\alpha$ weakly in $L^2(Q)$, it holds

$$\lim_\delta \delta^{-1}\langle G, u^\delta\rangle_{\mathbb{R}^3} = -2\,\langle \Sigma_{\alpha 3}, \chi_\alpha\rangle_{\mathbb{R}^3} \ . \tag{3.11}$$

Proof. See Appendix.

In view of Lemma 3.3, we are led to establish compactness properties for sequences u^δ such that the L^2-norm of $e^\delta(u^\delta)$ is uniformly bounded.
Before stating these compactness properties, which are summarized in the next proposition, we need to introduce a shape potential ψ_D associated to the section D, defined as the unique solution of

$$-\Delta\psi_D = 2 \quad , \quad \psi_D \in H^1_0(D).$$

Some properties of this function, well known in classical torsion theory, are recalled in Lemma 3.5.

Proposition 3.4 *Let $u^\delta \in \mathcal{C}^\infty(Q;\mathbb{R}^3)$ be a sequence with*

$$\int_Q u^\delta\, dx = \int_Q \psi_D \operatorname{curl} u^\delta\, dx = 0 \qquad \forall\delta \ . \tag{3.12}$$

Assume that $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$ and $\lim_{\delta\to 0} e^\delta_{i3}(u^\delta) = \chi_i$ weakly in $L^2(Q)$. Then, up to subsequences,

(i) there exists $\overline{u} \in BN(Q)$ such that $\lim_{\delta\to 0} u^\delta = \overline{u}$ weakly in $L^2(Q;\mathbb{R}^3)$;

(ii) setting

$$v^\delta_\alpha := \delta^{-1}(u^\delta - \overline{u})_\alpha - \delta^{-1}|D|^{-1}[[u^\delta - \overline{u}]]_\alpha$$

$$v^\delta_3 := \delta^{-1}(u^\delta - \overline{u})_3 - \delta^{-1}|D|^{-1}\Big([[u^\delta - \overline{u}]]_3 - x_\alpha[[u^\delta - \overline{u}]]'_\alpha\Big) \ ,$$

there exist $c \in H^1_m(I)$ and $w \in L^2(I;H^1_m(D))$ such that

$$\lim_{\delta\to 0}(v^\delta_1, v^\delta_2) = c(x_3)(-x_2, x_1) \ \text{weakly in } L^2(Q;\mathbb{R}^2)$$

$$\lim_{\delta\to 0} v^\delta_3 = w \ \text{weakly in } H^{-1}(I;L^2(D)) \ ;$$

(iii) $(\chi_1,\chi_2) = \frac{1}{2}\big(c'(x_3)(-x_2,x_1) + \nabla_{x'}w\big)$ in $L^2(Q;\mathbb{R}^2)$ and $\chi_3 = e_{33}(\overline{u})$ in $L^2(Q)$.

For the proof of Proposition 3.4 we need some preliminary lemmas.

Lemma 3.5 *The potential* ψ_D *is positive in* D. *Moreover, setting*

$$\gamma := \int_D |\nabla \psi_D|^2 \, dx' = 2 \int_D \psi_D \, dx' \ , \tag{3.13}$$

there hold

$$\inf \Big\{ \int_D |\nabla \psi|^2 \, dx' \ : \ \psi \in \mathcal{C}_0^\infty(D) \ , \ \int_D \psi \, dx' = 1 \Big\} = 4\gamma^{-1} \tag{3.14}$$

and

$$\inf \Big\{ \int_D |(-x_2, x_1) + \nabla w|^2 \, dx' \ : \ w \in H^1(D) \Big\} = \gamma \ . \tag{3.15}$$

Proof. See Appendix.

Lemma 3.6 *There exists positive constants* $C = C(D)$ *such that, for every* $v \in H^1_m(D;\mathbb{R}^2)$*, it holds*

$$\|v\|_{L^2(D;\mathbb{R}^2)} \leq C\Big(\|e(v)\|_{L^2(D;\mathbb{R}^{2\times 2}_{\rm sym})} + \Big| \int_D (\nabla \psi_D \wedge v) \, dx' \Big| \Big) \tag{3.16}$$

$$\| \int_D (\nabla \psi_D \wedge v) \, dx' - \operatorname{curl} v \|_{L^2(D)} \leq C \|e(v)\|_{L^2(D;\mathbb{R}^{2\times 2}_{\rm sym})} \ . \tag{3.17}$$

Proof. See Appendix.

Lemma 3.7 *Let* u^δ *be a sequence in* $\mathcal{C}^\infty(Q;\mathbb{R}^3)$ *with* $e^\delta(u^\delta)$ *bounded in* $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$ *and such that, for every* δ*, it holds:*

$$\int_Q \psi_D \operatorname{curl}_{x'}(u_1^\delta, u_2^\delta) \, dx = 0 \ . \tag{3.18}$$

Then the sequence

$$c^\delta(x_3) := \frac{1}{2\delta} \int_D \big(\nabla \psi_D \wedge (u_1^\delta, u_2^\delta) \big) \, dx' \ , \tag{3.19}$$

turns out to be bounded in $H^1(I)$.

Proof. See Appendix.

We can now give the

Proof of Proposition 3.4

For convenience, the proof is divided into several steps.

Step 1. The sequence $\int_Q \operatorname{curl} u^\delta \, dx$ *is bounded in* $\mathbb{R}^3$.

A version of the Korn inequality (see (28) in [25]) states that the skew symmetric part $\nabla^a u$ of the gradient satisfies

$$\int_Q \Big| \nabla^a u - \Big(\frac{1}{|Q|} \int \nabla^a u \Big) \Big|^2 dx \leq C \int_Q |e(u)|^2 \, dx \qquad \forall u \in H^1(Q;\mathbb{R}^3) \ . \tag{3.20}$$

We apply such inequality to

$$\tilde{u}^\delta := u^\delta - \frac{1}{2} b^\delta \wedge x \, , \quad \text{with } b^\delta := \frac{1}{|Q|} \int_Q \operatorname{curl} u^\delta \, dx \, .$$

By definition $\int_Q \operatorname{curl} \tilde{u}^\delta \, dx = 0$ and $e(\tilde{u}^\delta) = e(u^\delta)$, moreover by assumption $e(u^\delta)$ is bounded in $L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym})$, then by (3.20) we deduce that

$$\| \operatorname{curl} \tilde{u}^\delta \|_{L^2(Q:\mathbb{R}^3)} \le C \,. \tag{3.21}$$

We now exploit the hypothesis (3.12): since $\operatorname{curl} u^\delta = \operatorname{curl} \tilde{u}^\delta + b^\delta$, for every δ we have

$$\int_Q \psi_{_D} \operatorname{curl} \tilde{u}^\delta \, dx + b^\delta \int_D \psi_{_D} \, dx' = 0 \,,$$

that is, recalling the definition (3.13) of γ,

$$\frac{\gamma}{2} b^\delta = - \int_Q \psi_{_D} \operatorname{curl} \tilde{u}^\delta \, dx \,.$$

Thanks to (3.21) the right hand side is bounded , then we conclude that b^δ is bounded in $\mathbb{R}^3$.

Step 2. The sequence u^δ is bounded in $H^1(Q;\mathbb{R}^3)$ and any weak limit belongs to $BN(Q)$.
Applying the Korn inequality (3.20) to the sequence u^δ and taking into account that $\int_Q \operatorname{curl} u^\delta \, dx$ is bounded as shown in Step 1, we deduce that the L^2-norm of ∇u^δ remains bounded. Since we also know that $\int_Q u^\delta \, dx = 0$, the Poincaré-Wirtinger inequality ensures that the sequence u^δ is bounded, and hence weakly precompact, in $H^1(Q;\mathbb{R}^3)$. Again by the L^2-boundedness of $e^\delta(u^\delta)$, any weak L^2-limit $\overline{u}$ of u^δ satisfies $e_{ij}(\overline{u}) = 0$ for all $(i,j) \neq (3,3)$, and hence it belongs to $BN(Q)$. Moreover, we observe that the two integral conditions (3.12) hold also for the limit $\overline{u}$, then one can easily deduce that the Bernoulli-Navier field $\overline{u}$ is of the form (2.1).
Finally, taking the weak L^2-limit of the sequence $e_{33}(u^\delta)$, one obtains immediately that χ_3 agrees with $e_{33}(\overline{u})$.

Step 3. The sequence v^δ_α is bounded in $L^2(Q;\mathbb{R}^2)$.
Let us apply Lemma 3.6 to the sequence $v^\delta_\alpha(\cdot, x_3)$ for fixed x_3 (notice that $v^\delta_\alpha(\cdot, x_3)$ is indeed in $H^1_m(D;\mathbb{R}^2)$). By taking into account that $e_{\alpha\beta}(\overline{u}) = 0$ and $\int_D (\nabla \psi_{_D} \wedge (\overline{u}_1, \overline{u}_2)) \, dx' = 0$ (since $\overline{u}$ is of the form (2.1)), we deduce

$$\int_D |(v^\delta_1, v^\delta_2)|^2 \, dx' \le C \Big[\frac{1}{\delta^2} \int_D |e_{\alpha\beta}(u^\delta)|^2 \, dx' + \Big| \frac{1}{\delta} \int_D \big(\nabla \psi_{_D} \wedge (u^\delta_1, u^\delta_2) \big) \, dx' \Big|^2 \Big] \qquad \text{for a.e. } x_3 \in I \,.$$

Then, integrating with respect to x_3 over I, we get

$$\int_Q |(v^\delta_1, v^\delta_2)|^2 \, dx \le C \Big[\delta^2 \int_Q |e^\delta_{\alpha\beta}(u^\delta)|^2 \, dx + \int_I |2c^\delta(x_3)|^2 \, dx_3 \Big] \,,$$

where the sequence c^δ is associated to the sequence u^δ according to formula (3.19). Since the sequence u^δ satisfies $e^\delta(u^\delta)$ bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$ and condition (3.18), Lemma 3.7 allows to conclude that v^δ_α is bounded in $L^2(Q;\mathbb{R}^2)$.

Step 4. Any weak limit of (v^δ_1, v^δ_2) is of the form $c(x_3)(-x_2, x_1)$, for some $c \in L^2_m(I)$.
Since $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$, there exists a positive constant C such that $\|e_{\alpha\beta}(v^\delta)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})} \le C\delta$. Therefore any weak limit $\overline{v} = (\overline{v}_1, \overline{v}_2)$ satisfies $e_{\alpha\beta}(\overline{v}) = 0$, and hence it is of the form $(\overline{v}_1, \overline{v}_2) = c(x_3)(-x_2, x_1) + (d_1(x_3), d_2(x_3))$ for some c and d_α in $L^2(I)$. Since by their

definition v_α^δ satisfy $[[v_\alpha^\delta]]=0$, we have also $[[\overline{v}_\alpha]]=0$, so that $d_\alpha=0$. It remains to prove that c has zero integral mean. Set

$$\omega^\delta := \frac{1}{2\delta}\big(\partial_1 u_2^\delta - \partial_2 u_1^\delta\big) = \frac{1}{2}\big(\partial_1 v_2^\delta - \partial_2 v_1^\delta\big) \ .$$

We observe that, since by assumption $\int_Q \psi_{_D} \operatorname{curl} u^\delta\, dx = 0$, the functions ω^δ satisfy

$$\int_Q \psi_{_D}\, \omega^\delta\, dx = 0 \qquad \forall \delta \ . \tag{3.22}$$

Since $\lim_{\delta\to 0}\omega^\delta = c(x_3)$ in $\mathcal{D}'(Q)$, and since by definition the sequence ω^δ remains bounded in $L^2(I;H^{-1}(D))$, we have also $\lim_{\delta\to 0}\omega^\delta = c$ weakly in $L^2(I;H^{-1}(D))$. In particular, taking as a test funcion $\psi_{_D}$, passing to the limit as $\delta\to 0$ in (3.22), we obtain $\int_I c(x_3)\,dx_3 = 0$.

Step 5. The distributional derivative of c is given by $c'(x_3) = \partial_1\chi_2 - \partial_2\chi_1$.
Since (v_1^δ, v_2^δ) converges to $(\overline{v}_1,\overline{v}_2)$ weakly in $L^2(Q;\mathbb{R}^2)$, it holds

$$\lim_{\delta\to 0}\partial_3\omega^\delta = \partial_3\big[\frac{1}{2}\big(\partial_1\overline{v}_2 - \partial_2\overline{v}_1\big)\big] = c'(x_3) \qquad \text{in } \mathcal{D}'(Q) \ .$$

On the other hand, since

$$\partial_3\omega^\delta = \frac{1}{\delta}\big[\partial_1 e_{23}(u^\delta) - \partial_2 e_{13}(u^\delta)\big] - \frac{1}{2\delta}\big(\partial_1\partial_2 u_3^\delta - \partial_2\partial_1 u_3^\delta\big) = \partial_1 e_{23}^\delta(u^\delta) - \partial_2 e_{13}^\delta(u^\delta) \ ,$$

it also holds

$$\lim_{\delta\to 0}\partial_3\omega^\delta = \partial_1\chi_2 - \partial_2\chi_1 \qquad \text{in } \mathcal{D}'(Q) \ .$$

It follows that $\partial_1\chi_2 - \partial_2\chi_1 = c'(x_3)$ in $\mathcal{D}'(Q)$.

Step 6. The function c belongs to $H^1_m(I)$.
Let us fix $\varphi\in\mathcal{C}_0^\infty(I)$, and $\psi\in\mathcal{C}_0^\infty(D)$ with $\int_D \psi\, dx' = 1$. We have

$$\langle \partial_1\chi_2 - \partial_2\chi_1, \varphi(x_3)\psi(x')\rangle_{\mathbb{R}^3} = \int_Q (\chi_1\partial_2\psi - \chi_2\partial_1\psi)\varphi\, dx \le \frac{1}{2}\Big(\int_Q |\chi|^2\,dx + \int_D |\nabla\psi|^2\,dx' \int_I \varphi^2\,dx_3\Big) \ . \tag{3.23}$$

By Step 3, we know that

$$\langle \partial_1\chi_2 - \partial_2\chi_1, \varphi(x_3)\psi(x')\rangle_{\mathbb{R}^3} = \int_I c'(x_3)\varphi(x_3)\,dx_3 \ , \tag{3.24}$$

Combining (3.23) and (3.24), we obtain

$$\int_I c'(x_3)\varphi(x_3)\,dx_3 - \frac{1}{2}\int_D |\nabla\psi|^2\,dx' \int_I \varphi^2\,dx_3 \le \frac{1}{2}\int_Q |\chi|^2\,dx \ .$$

By the Fenchel inequality, this implies

$$\int_I |c'(x_3)|^2\,dx_3 \le \Big(\int_D |\nabla\psi|^2(x')\,dx'\Big)\Big(\int_Q |\chi|^2\,dx\Big) \ .$$

Passing to the infimum over all the functions ψ in $\mathcal{C}_0^\infty(D)$ with $\int_D \psi\, dx' = 1$, and applying (3.14) in Lemma 3.5, we obtain

$$\int_I |c'(x_3)|^2\, dx_3 \le 4\gamma^{-1} \int_Q |\chi|^2\, dx\ ,$$

where γ is the positive constant defined in (3.13).

Step 7. The sequence v_3^δ converges weakly in $H^{-1}(I; L^2(D))$ to some limit w.
A partial Korn's inequality proved in [22] states that, for any $z \in H^1(Q;\mathbb{R}^3)$, it holds

$$\left\| z_3 - |D|^{-1}\left([[z_3]] - x_\alpha\, [[z_\alpha]]'\right)\right\|_{H^{-1}(I;L^2(D))} \le C\, \left(\|e_{\alpha\beta}(z)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})} + \|e_{\alpha 3}(z)\|_{L^2(Q;\mathbb{R}^2)} \right)\ .$$

Applying this inequality to the sequence $z^\delta := \delta^{-1}(u^\delta - \overline{u})$, since by assumption $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$ and $\overline{u} \in BN(Q)$, we deduce that v_3^δ is bounded in $H^{-1}(I;L^2(D))$. Therefore there exists w such that $\lim_{\delta\to 0} v_3^\delta = w$ weakly in $H^{-1}(I;L^2(D))$. Notice that, since $\mathcal{D}(Q) \subset H^1_0(I;L^2(D))$, the convergence holds also in $\mathcal{D}'(Q)$.

Step 8. It holds $(\chi_1,\chi_2) = \frac{1}{2}\big(c'(x_3)(-x_2,x_1) + \nabla_{x'} w\big)$ in $L^2(Q;\mathbb{R}^2)$ and $w \in L^2(I;H^1_m(D))$.
Since

$$u_\alpha^\delta = \overline{u}_\alpha + \delta v_\alpha^\delta + |D|^{-1}[[u^\delta - \overline{u}]]_\alpha$$

$$u_3^\delta = \overline{u}_3 + \delta v_3^\delta + |D|^{-1}\Big([[u^\delta - \overline{u}]]_3 - x_\alpha [[u^\delta - \overline{u}]]'_\alpha\Big)\ ,$$

we have $e^\delta_{\alpha 3}(u^\delta) = e_{\alpha 3}(v^\delta)$. We know by assumption that $\lim_{\delta\to 0} e^\delta_{\alpha 3}(u^\delta) = \chi_\alpha$ weakly in $L^2(Q)$, and by Steps 4 and 7 that $\lim_{\delta\to 0}(e_{13}(v^\delta), e_{23}(v^\delta)) = \frac{1}{2}\big(c'(x_3)(-x_2,x_1) + \nabla_{x'} w\big)$ in $\mathcal{D}'(Q;\mathbb{R}^2)$. We infer that the equality $(\chi_1,\chi_2) = \frac{1}{2}\big(c'(x_3)(-x_2,x_1) + \nabla_{x'} w\big)$ holds in $\mathcal{D}'(Q;\mathbb{R}^2)$. This implies that $\nabla_{x'} w \in L^2(Q;\mathbb{R}^2)$ (because $\chi_\alpha \in L^2(Q)$ and by Step 6 $c' \in L^2(I)$), and that the same equality remains true in $L^2(Q;\mathbb{R}^2)$.
Finally we notice that by construction $[[v_3^\delta]] = 0$ for each δ, so that also $[[w]] = 0$. Therefore, applying Poincaré-Wirtinger inequality section by section we infer that $w \in L^2(I;H^1_m(D))$. $\square$

3.2 Part II: asymptotics of fictitious problems

Proposition 3.8 *As $\delta \to 0$, the sequence $\widetilde{\mathcal{C}}^\delta(\theta)$ in (3.9) Γ-converges, in the weak * topology of $L^\infty(Q;[0,1])$, to the limit compliance $\mathcal{C}^{lim}(\theta)$ defined by (3.6). Hence the sequence $\widetilde{\phi}^\delta(k)$ in (3.8) converges to the limit $\phi(k)$ defined by (3.5).*

Proof. By definition of Γ-convergence, the statement means that the so-called Γ-liminf and Γ-limsup inequalities hold:

$$\inf\Big\{ \liminf \widetilde{\mathcal{C}}^\delta(\theta^\delta)\ :\ \theta^\delta \stackrel{*}{\rightharpoonup} \theta \Big\} \ge \mathcal{C}^{lim}(\theta) \tag{3.25}$$

$$\inf\Big\{ \limsup \widetilde{\mathcal{C}}^\delta(\theta^\delta)\ :\ \theta^\delta \stackrel{*}{\rightharpoonup} \theta \Big\} \le \mathcal{C}^{lim}(\theta)\ . \tag{3.26}$$

Proof of (3.25). Consider an arbitrary sequence $\theta^\delta \overset{*}{\rightharpoonup} \theta$. We claim that, for every $v \in TW(Q)$, it holds

$$\lim_{\delta\to 0} \int_Q j(e^\delta(\delta v))\, \theta^\delta \, dx = \int_Q \overline{j}(e_{13}(v), e_{23}(v), 0)\, \theta \, dx \tag{3.27}$$

Once proved this claim, (3.25) follows immediately. Indeed, it is enough to take a sequence $v_k \in TW(Q)$ such that

$$\mathcal{C}^{lim}(\theta) = \lim_k \left\{ \langle G, v_k \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v_k), e_{23}(v_k), 0)\, \theta \, dx \right\} .$$

Applying (3.27) to each v_k, and setting $v_k^\delta := \delta v_k$, we get

$$\mathcal{C}^{lim}(\theta) = \lim_k \lim_\delta \left\{ \delta^{-1} \langle G, v_k^\delta \rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(v_k^\delta))\, \theta^\delta \, dx \right\} \leq \liminf_\delta \widetilde{\mathcal{C}}^\delta(\theta^\delta) .$$

To prove (3.27), we notice that, for every $v \in TW(Q)$,

$$e^\delta(\delta v) \to \begin{bmatrix} 0 & e_{\alpha 3}(v) \\ e_{\alpha 3}(v) & 0 \end{bmatrix} \qquad \text{a.e. on } Q .$$

Moreover, an easy algebraic calculation shows that

$$\overline{j}\big(e_{13}(v), e_{23}(v), 0\big) = j \begin{bmatrix} 0 & e_{\alpha 3}(v) \\ e_{\alpha 3}(v) & 0 \end{bmatrix} .$$

Then, by dominated convergence, we have $j(e^\delta(\delta v)) \to \overline{j}(e_{13}(v), e_{23}(v), 0)$ strongly in $L^1(Q)$. Therefore, recalling that by assumption $\theta^\delta \overset{*}{\rightharpoonup} \theta$, the integrand in the left hand side of (3.27) is the product between a strongly and a weakly convergent sequence, and we deduce that (3.27) holds.

Proof of (3.26). We have to find a recovery sequence $\theta^\delta \overset{*}{\rightharpoonup} \theta$ such that $\limsup_\delta \widetilde{\mathcal{C}}^\delta(\theta^\delta) \leq \mathcal{C}^{lim}(\theta)$. Let us first show that, under the assumption $\inf_Q \theta > 0$, we are done simply by taking $\theta^\delta \equiv \theta$. Let u^δ be a sequence of functions satisfying

$$\limsup_\delta \widetilde{\mathcal{C}}^\delta(\theta) = \limsup_\delta \left\{ \delta^{-1} \langle G, u^\delta \rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u^\delta))\, \theta \, dx \right\} . \tag{3.28}$$

Since we may assume with no loss of generality that $\limsup_\delta \widetilde{\mathcal{C}}^\delta(\theta^\delta) > -\infty$, and since by assumption θ is bounded from below, we are in a position to apply Lemma 3.3. Then, the sequence $e^\delta(u^\delta)$ is bounded in $L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym})$. Denoting by χ_α the weak L^2-limit of $e^\delta_{\alpha 3}(u^\delta)$, Lemma 3.3 also gives

$$\lim_{\delta\to 0} \delta^{-1} \langle G, u^\delta \rangle_{\mathbb{R}^3} = -2 \, \langle \Sigma_{\alpha 3}, \chi_\alpha \rangle_{\mathbb{R}^3} . \tag{3.29}$$

Next we notice that the sequence u^δ satisfies also the assumptions of Proposition 3.4. Indeed, the conditions in (3.12) hold up to subtracting from u^δ the rigid motion $a^\delta + b^\delta \wedge x$, with

$$a^\delta := \frac{1}{|Q|} \int_Q u^\delta \, dx , \qquad b^\delta := \frac{1}{2|Q|} \int_Q \psi_D \operatorname{curl} u^\delta \, dx .$$

Thus, if c and w are associated to the sequence u^δ as in Proposition 3.4 (ii), we may write $\chi_\alpha = e_{\alpha 3}(v)$, with $v := (-c(x_3)x_2, c(x_3)x_1, w) \in TW(Q)$. Combining this equality with (3.29) we obtain

$$\lim_{\delta \to 0} \delta^{-1} \langle G, u^\delta \rangle_{\mathbb{R}^3} = -2 \, \langle \Sigma_{\alpha 3}, e_{\alpha 3}(v) \rangle_{\mathbb{R}^3} = -\langle \Sigma, e(v) \rangle_{\mathbb{R}^3} = \langle G, v \rangle_{\mathbb{R}^3} \ . \tag{3.30}$$

We now turn attention to estimate from below $\int_Q j(e^\delta(u^\delta))\, \theta \, dx$. We claim that

$$\liminf_{\delta \to 0} \int_Q j(e^\delta(u^\delta))\, \theta \, dx \geq \int_Q \overline{j}(\chi_1, \chi_2, \chi_3)\, \theta \, dx \tag{3.31}$$

(where χ_i is the weak L^2-limit of $e^\delta_{i3}(u^\delta)$).
Indeed, for every $\xi \in \mathbb{R}^3$, let us denote by $E_0 \xi$ the symmetric matrix

$$E_0 \xi := \frac{1}{2} \sum_{i=1}^{3} \xi_i (e_i \otimes e_3 + e_3 \otimes e_i) \ . \tag{3.32}$$

The Fenchel inequality and the weak L^2-convergence of $e^\delta_{i3}(u^\delta)$ to χ_i yield

$$\begin{aligned} \liminf_\delta \int_Q j(e^\delta(u^\delta))\theta \, dx \quad &\geq \liminf_\delta \Big\{ \int_Q e^\delta(u^\delta) \cdot E_0\xi \, \theta \, dx - \int_Q j^*(E_0 \xi)\, \theta \, dx \Big\} \\ &= \int_Q (\chi_1, \chi_2, \chi_3) \cdot \xi \, \theta \, dx - \int_Q j^*(E_0\xi)\, \theta \, dx \end{aligned}$$

for every $\xi \in L^2(Q; \mathbb{R}^3)$.
By using the definition of $\overline{j}$, one can easily check that

$$\overline{j}^{\,*}(\xi) = j^*(E_0 \xi) \qquad \forall \xi \in \mathbb{R}^3 \ . \tag{3.33}$$

Such identity and the arbitrariness of $\xi \in L^2(Q; \mathbb{R}^3)$ in the previous inequality yield

$$\liminf_\delta \int_Q j(e^\delta(u^\delta))\, \theta \, dx \geq \sup_\xi \Big\{ \int_Q (\chi_1, \chi_2, \chi_3) \cdot \xi \, \theta \, dx - \int_Q \overline{j}^{\,*}(\xi)\, \theta \, dx \Big\} \ .$$

By passing to the supremum over $\xi \in L^2(Q; \mathbb{R}^3)$ under the sign of integral (see *e.g.* [6, Lemma A.2]), and taking into account that $\overline{j} = \overline{j}^{\,**}$, we get the required inequality (3.31). Finally, by the definition of v, we have

$$\int_Q \overline{j}(\chi_1, \chi_2, \chi_3)\, \theta \, dx = \int_Q \overline{j}\big(e_{\alpha 1}(v), e_{\alpha 2}(v), \chi_3\big)\, \theta \, dx \geq \int_Q \overline{j}\big(e_{\alpha 1}(v), e_{\alpha 2}(v), 0\big)\, \theta \, dx \ . \tag{3.34}$$

From (3.28), (3.30), (3.31) and (3.34), recalling the expression (3.6) of $\mathcal{C}^{lim}(\theta)$, it follows that $\limsup_\delta \widetilde{\mathcal{C}}^\delta(\theta^\delta) \leq \mathcal{C}^{lim}(\theta)$. It remains to get rid of the additional assumption $\inf_Q \theta > 0$. This can be done via a standard density argument. Indeed, for any θ we may find a sequence θ^h with $\inf_Q \theta^h > 0$ such that $\theta^h \overset{*}{\rightharpoonup} \theta$. Then, since the left hand side of (3.26) (usually called $\Gamma - \limsup \widetilde{\mathcal{C}}^\delta(\theta)$), is weakly * lower semicontinuous, and $\mathcal{C}^{lim}(\theta)$ is weakly * continuous, we obtain

$$(\Gamma - \limsup_\delta \widetilde{\mathcal{C}}^\delta)(\theta) \leq \liminf_h (\Gamma - \limsup_\delta \widetilde{\mathcal{C}}^\delta)(\theta^h) \leq \lim_h \mathcal{C}^{lim}(\theta^h) = \mathcal{C}^{lim}(\theta) \ .$$

The convergence of $\widetilde{\phi}^\delta(k)$ to $\phi(k)$ follows immediately by well-known properties of Γ-convergence. $\square$

3.3 Part III: back to the initial problems

In order to obtain the asymptotics of the original problems $\phi^\delta(k)$ defined in (3.1), we will bound them both from above and from below in terms of fictitious problems which admit the same limit. We first remark that, for every k, it holds

$$\phi^\delta(k) = \inf \left\{ \overline{\mathcal{C}}^\delta(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\},$$

being $\overline{\mathcal{C}}^\delta(\theta)$ the lower semicontinuous envelope, in the weak * topology of $L^\infty(Q;[0,1])$, of the functional which is defined as in (3.2) if θ is the characteristic function of a set ω, and $+\infty$ otherwise. Then, by the weak * lower semicontinuity of the fictitious compliance defined in (3.9), we immediately obtain the inequality

$$\widetilde{\mathcal{C}}^\delta(\theta) \leq \overline{\mathcal{C}}^\delta(\theta) \qquad \forall \theta \in L^\infty(Q;[0,1]),$$

and hence the following lower bound for $\phi^\delta(k)$:

$$\widetilde{\phi}^\delta(k) \leq \phi^\delta(k). \tag{3.35}$$

On the other hand, finding an upper bound for $\phi^\delta(k)$ is a quite delicate problem, which has been treated in [10, Section 2.3]. For the benefit of the reader, let us briefly sketch an outline of such upper bound. Let $j_0 : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$ denote the modified stored energy density defined by

$$j_0(z) := \sup \left\{ z \cdot \xi - j^*(\xi) \ : \xi \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \ , \ \det(\xi) = 0 \right\}. \tag{3.36}$$

The potential j_0 plays an important role in the problem of minimizing the compliance for small volume fractions: heuristically, the condition $\det \xi = 0$ appearing in (3.36) corresponds to the degeneracy of stress tensors occurring when the material concentrates on low-dimensional sets (see [4, 8, 10] for more details, and also [2] for the explicit computation of j_0^*).
The main properties of j_0 are summarized in the next lemma, where $\overline{j_0}$ denotes the $2d$ reduced counterpart of j_0, defined as in (3.3) with j replaced by j_0.

Lemma 3.9 *The function j_0 satisfies $j_0 \leq j$, is coercive and homogeneous of degree 2. Moreover, the following algebraic identity holds*

$$\overline{j_0}(y) = \overline{j}(y) \qquad \forall y \in \mathbb{R}^3. \tag{3.37}$$

Proof. See Appendix.

Let us consider on $L^\infty(Q;[0,1])$ the compliance functional associated with j_0

$$\widetilde{\mathcal{C}}_0^\delta(\theta) := \sup \left\{ \delta^{-1} \langle G, u \rangle_{\mathbb{R}^3} - \int_Q j_0(e^\delta(u))\, \theta \, dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\}, \tag{3.38}$$

and the corresponding fictitious problems

$$\widetilde{\phi}_0^\delta(k) := \inf \left\{ \widetilde{\mathcal{C}}_0^\delta(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\}. \tag{3.39}$$

Under the assumption (2.5) on the load, by applying [10, Proposition 2.8], we deduce the following crucial estimate:

$$\overline{\mathcal{C}}^\delta(\theta) \le \widetilde{\mathcal{C}}_0^\delta(\theta) \qquad \forall \theta \in L^\infty(Q;[0,1])\,.$$

Consequently, as a counterpart to (3.35), one obtains the upper bound

$$\phi^\delta(k) \le \widetilde{\phi}_0^\delta(k)\,. \tag{3.40}$$

We can now give the

Proof of Theorem 3.2

We first prove that the sequence $\widetilde{\mathcal{C}}_0^\delta(\theta)$ defined in (3.38) Γ-converges, in the weak * topology of $L^\infty(Q;[0,1])$, to the limit compliance $\mathcal{C}^{lim}(\theta)$ defined by (3.6). Indeed, exploiting the coercivity and homogeneity of j_0 (*cf.* Lemma 3.9), the same proof used for Proposition 3.8 is valid, and gives the Γ-convergence of $\widetilde{\mathcal{C}}_0^\delta(\theta)$ toward the functional

$$\sup\left\{\langle G,v\rangle_{\mathbb{R}^3} - \int_Q \overline{j_0}\big(e_{13}(v),e_{23}(v),0\big)\,\theta\,dx \;:\; v\in TW(Q)\right\}.$$

Since by Lemma 3.9 $\overline{j_0} = \overline{j}$, the Γ-limit above agrees with $\mathcal{C}^{lim}(\theta)$. As a consequence the fictitious problems $\widetilde{\phi}_0^\delta(k)$ defined in (3.39) converge to $\phi(k)$.

Combining this result with the one obtained in Proposition 3.8, thanks to the estimates (3.35) and (3.40), we infer that also the sequence $\phi^\delta(k)$ converges to $\phi(k)$.

Let $\omega^\delta \subset Q$ be a sequence of domains such that $\phi^\delta(k) = \mathcal{C}^\delta(\omega^\delta) + k|\omega^\delta| + o(1)$. Since we know that the sequences $\widetilde{\phi}^\delta(k)$ and $\phi^\delta(k)$ have the same limit as $\delta \to 0$, we deduce that $\widetilde{\phi}^\delta(k) = \widetilde{\mathcal{C}}^\delta(\mathbb{1}_{\omega^\delta}) + k\int_Q \mathbb{1}_{\omega^\delta}\,dx + o(1)$. Since by Proposition 3.8 the sequence $\widetilde{\mathcal{C}}^\delta(\theta) + k\int_Q \theta\,dx$ Γ-converges to $\mathcal{C}^{lim}(\theta) + k\int_Q \theta\,dx$ in the the weak * topology $L^\infty(Q;[0,1])$, any cluster point of $\mathbb{1}_{\omega^\delta}$ is a solution $\overline{\theta}$ to problem (3.5).

It remains to show that the limit compliance $\mathcal{C}^{lim}(\theta)$ defined in (3.6) may be also rewritten as in (3.7). To this end, it is enough to write any $v \in TW(Q)$ under the form (2.2), and exploit the identities (2.6) and (3.4).

□

4 Equivalent formulations of $\phi(k)$ and optimality conditions

In view of Theorem 3.2, the limit problem (3.5) we have to solve is a $3d$ variational problem for material densities θ in $L^\infty(Q;[0,1])$. We are now going to show that equivalent formulations for $\phi(k)$ can be obtained dealing either with displacement fields $v \in TW(Q)$ (see Theorem 4.1) or with shear stress components $\sigma \in L^2(Q;\mathbb{R}^2)$ (see Theorem 4.2). These different formulations will allow us to write down necessary and sufficient optimality conditions in term of optimal triples $(\overline{\theta},\overline{v},\overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times L^2(Q;\mathbb{R}^2)$ (see Theorem 4.5).

Theorem 4.1 *For every $k \in \mathbb{R}$, it holds*

$$\phi(k) = \sup\left\{\langle G,v\rangle_{\mathbb{R}^3} - \int_Q \left[\overline{j}\big(e_{13}(v),e_{23}(v),0\big) - k\right]_+ dx \;:\; v\in TW(Q)\right\} \tag{4.1}$$

$$= \sup \Big\{ \langle m_G, c\rangle_{\mathbb{R}} + \langle G_3, w\rangle_{\mathbb{R}^3} - \int_Q \Big[\frac{\eta}{2}\left|c'(x_3)(-x_2,x_1) + \nabla_{x'} w\right|^2 - k\Big]_+ dx : \\ c \in H^1_m(I), w \in L^2(I; H^1_m(D))\Big\}.$$

Proof. Let $X = L^\infty(Q;[0,1])$ endowed with the weak * topology, and $Y = H^1(Q;\mathbb{R}^3)$ endowed with the weak topology. On the product space $X \times Y$ we consider, for a fixed $k \in \mathbb{R}$, the Lagrangian

$$\mathcal{L}_k(\theta, v) := \begin{cases} \langle G, v\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j}\big(e_{13}(v), e_{23}(v), 0\big) - k\big]\theta\, dx & \text{if } v \in TW(Q) \\ -\infty & \text{otherwise .} \end{cases}$$

Since $\mathcal{L}_k(\theta, v)$ is convex in θ on the compact space X and concave in v on Y, the equality $\inf_X \sup_Y \mathcal{L} = \sup_Y \inf_X \mathcal{L}$ holds by a standard commutation argument, see for instance [26, Proposition A.8]. □

We now give the dual form of the displacement problem (4.1). We complement it with the dual form of the limit compliance $\mathcal{C}^{lim}(\theta)$ in (3.6), since this will be useful in writing the optimality conditions. Below, we denote by M_G a primitive of m_G in the sense of distributions:

$$M_G(x_3) := \int_{-\infty}^{x_3} m_G(s)\, ds \ .$$

Moreover, we denote by φ_k the function of one real variable given by

$$\varphi_k(s) := \begin{cases} \frac{1}{8\eta}s^2 + k & \text{if } |s| \geq \sqrt{8\eta k} \\ \sqrt{\frac{k}{2\eta}}|s| & \text{if } |s| \leq \sqrt{8\eta k}. \end{cases}$$

We point out that, for any $\xi = (\xi_1, \xi_2, 0)$, $\varphi_k(|\xi|)$ is the Fenchel conjugate of $[\overline{j}(y) - k]_+$. Indeed, by [10, Lemma 4.4], $[\overline{j}(y) - k]_+^*$ coincides with the convex envelope of the function $g_k : \mathbb{R}^3 \to \mathbb{R}$ defined as

$$g_k(\xi) = \begin{cases} \overline{j}^{\,*}(\xi) + k & \text{if } \xi \neq 0 \\ 0 \ . & \text{otherwise} \end{cases} \tag{4.2}$$

The explicit computation of such convex envelope at vectors $\xi \in \mathbb{R}^3$ of the kind $(\xi_1, \xi_2, 0)$, gives precisely $\varphi_k(|\xi|)$.

Theorem 4.2 *For every $\theta \in L^\infty(Q;[0,1])$ and every $k \in \mathbb{R}$, problems* (3.6) *and* (4.1) *admit respectively the dual formulations*

$$\mathcal{C}^{lim}(\theta) = \inf_{\sigma \in L^2(Q;\mathbb{R}^2)} \Big\{ \int_Q \theta^{-1} \overline{j}^{\,*}(\sigma_1, \sigma_2, 0)\, dx \ : \ \operatorname{div}_{x'} \sigma = -2\, G_3,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2M_G \Big\} \tag{4.3}$$

and

$$\phi(k) = \inf_{\sigma \in L^2(Q;\mathbb{R}^2)} \Big\{ \int_Q \varphi_k(|\sigma|)\, dx \ : \ \operatorname{div}_{x'} \sigma = -2\, G_3,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2M_G \Big\} \ . \tag{4.4}$$

Remark 4.3 *(link with the classical torsion problem).* Formulation (4.4) reveals that the limit optimization problem $\phi(k)$ can be solved section by section. Indeed, $\phi(k) = \int_I \Lambda_k(G_3, M_G)\, dx_3$, where, for any $q \in H^{-1}(D)$, $r \in \mathbb{R}$,

$$\Lambda_k(q,r) := \inf_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D \varphi_k(|\sigma|)\, dx' \,:\, \mathrm{div}_{x'} \sigma = -2q,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2r \right\} .$$

This way of computing $\phi(k)$ enlightens the link with the classical torsion problem. Actually, the compliance of a cylindrical rod of section $E \subset D$ under a torque r is classically written as

$$\inf_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D \frac{1}{8\eta} |\sigma|^2\, dx' \,:\, \mathrm{div}_{x'} \sigma = 0,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2r\,,\ \mathrm{spt}(\sigma) \subset \overline{E} \right\} . \tag{4.5}$$

The optimization of such compliance with respect to the domain E under the volume constraint $|E| = m$ reads

$$\inf_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D \frac{1}{8\eta} |\sigma|^2\, dx' \,:\, \mathrm{div}_{x'} \sigma = 0,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2r\,,\ |\mathrm{spt}(\sigma)| \leq m \right\} .$$

Introducing a Lagrange multiplier k, one is reduced to solve

$$\begin{aligned} &\inf_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D \frac{1}{8\eta} |\sigma|^2\, dx' + k|\mathrm{spt}(\sigma)| \,:\, \mathrm{div}_{x'} \sigma = 0,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2r \right\} \\ = &\inf_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D g_k(\sigma_1,\sigma_2,0)\, dx' \,:\, \mathrm{div}_{x'} \sigma = 0,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2r \right\} , \end{aligned}$$

being g_k the function defined in (4.2). The relaxed formulation of the latter problem is nothing else than $\Lambda_k(0,r)$. This concordance is somehow surprising, since formulation (4.5) is valid only for cylindrical rods (or rods with slowly varying section) whereas, in the formulation (1.3) of our initial optimization problems $\phi^\delta(k)$, no topological constraint is imposed on the admissible domains $\Omega \subset \delta\overline{D} \times I$. What can be inferred from this comparison is that optimal thin torsion rods searched in a very large class without imposing any geometrical restriction are in fact not sensibly different from the nearly cylindrical ones treated in the classical theory.

The proof of Theorem 4.2 is based on a standard convex duality lemma (see *e.g.* [5, Proposition 14]), that we recall for the benefit of the reader.

Lemma 4.4 *Let X, Y be Banach spaces. Let $A : X \to Y$ be a linear operator with dense domain $D(A)$. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, and $\Psi : Y \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous. Assume there exists $u_0 \in D(A)$ such that $\Phi(u_0) < +\infty$ and Ψ is continuous at $A(u_0)$. Let Y^* denote the dual space of Y, A^* the adjoint operator of A, and Φ^*, Ψ^* the Fenchel conjugates of Φ, Ψ. Then*

$$-\inf_{u \in X} \left\{ \Psi(Au) + \Phi(u) \right\} = \inf_{\sigma \in Y^*} \left\{ \Psi^*(\sigma) + \Phi^*(-A^*\sigma) \right\} ,$$

where the infimum on the right hand side is achieved.

Proof of Theorem 4.2

The dual form (4.3) of $\mathcal{C}^{lim}(\theta)$ is obtained by applying Lemma 4.4 with $X = TW(Q)$, $Y = L^2(Q;\mathbb{R}^2)$, $A(v) = (e_{13}(v), e_{23}(v))$, $\Phi(v) = -\langle G, v\rangle_{\mathbb{R}^3}$, and $\Psi(y) = \int_Q \bar{j}(y_1, y_2, 0)\theta\, dx$. By the same lemma applied with X, Y, A, and Φ as above, and $\Psi(y) = \int_Q [\bar{j}(y_1, y_2, 0) - k]_+\, dx$, one obtains

$$\phi(k) = \inf_{\sigma \in L^2(Q;\mathbb{R}^2)} \left\{ \int_Q [\bar{j} - k]_+^*(\sigma_1, \sigma_2, 0)\, dx \,:\, \mathrm{div}(E_0(\sigma_1, \sigma_2, 0)) + G \in TW(Q)^\perp \right\} .$$

Then equality (4.4) follows by taking into account that, at $\xi = (\xi_1, \xi_2, 0)$, the Fenchel conjugate of $[\bar{j}(y) - k]_+$ agrees with $\varphi_k(|\xi|)$, and that the constraint $\mathrm{div}(E_0(\sigma_1, \sigma_2, 0)) + G \in TW(Q)^\perp$ is equivalent to the two conditions $\mathrm{div}_{x'}\,\sigma = -2\, G_3$ and $[[x_1\sigma_2 - x_2\sigma_1]] = -2M_G$. □

Now, by using the equivalence between the different expressions for $\phi(k)$ given in Theorems 3.2, 4.1, and 4.2, we are able to provide necessary and sufficient optimality conditions. We say that $(\overline{\theta}, \overline{v}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times L^2(Q;\mathbb{R}^2)$ is an *optimal triple for* $\phi(k)$ if:

(·) $\overline{\theta}$ solves problem (3.5);

(·) $\overline{v}$ solves problem (4.1) and is optimal for $\mathcal{C}^{lim}(\overline{\theta})$ in its primal form (3.6);

(·) $\overline{\sigma}$ solves problem (4.4) and is optimal for $\mathcal{C}^{lim}(\overline{\theta})$ in its dual form (4.3).

Theorem 4.5 *A triple* $(\overline{\theta}, \overline{v}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times L^2(Q;\mathbb{R}^2)$ *is optimal for* $\phi(k)$ *if and only if it satisfies the following system:*

$$\mathrm{div}_{x'}\,\overline{\sigma} = -2\, G_3 \ , \quad [[x_1\overline{\sigma}_2 - x_2\overline{\sigma}_1]] = -2M_G \tag{4.6}$$

$$(\overline{\sigma}_1, \overline{\sigma}_2, 0) = \overline{\theta}\ \bar{j}\,'\big(e_{13}(\overline{v}), e_{23}(\overline{v}), 0\big) \tag{4.7}$$

$$(\overline{\sigma}_1, \overline{\sigma}_2, 0) \in \partial\Big([\bar{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - k]_+\Big) \tag{4.8}$$

$$\overline{\theta}\,\big(\bar{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - k\big) = [\bar{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - k]_+ \tag{4.9}$$

Proof. Firstly note that, for every $v \in TW(Q)$, and any field σ admissible in any of the dual problems (4.3) and (4.4), there holds:

$$\begin{aligned} \langle G, v\rangle_{\mathbb{R}^3} &= -\langle \mathrm{div}(E_0(\sigma_1, \sigma_2, 0)), v\rangle_{\mathbb{R}^3} \\ &= \int_Q E_0(\sigma_1, \sigma_2, 0)\cdot e(v)\, dx = \int_Q (\sigma_1, \sigma_2)\cdot(e_{13}(v), e_{23}(v))\, dx \ . \end{aligned} \tag{4.10}$$

Assume now that $(\overline{\theta}, \overline{v}, \overline{\sigma})$ is an optimal triple for $\phi(k)$. Then clearly (4.6) holds since $\overline{\sigma}$ must be admissible for problem (4.4). Moreover, since $\overline{\sigma}$ is optimal for the dual form (4.3) of $\mathcal{C}^{lim}(\overline{\theta})$, necessarily it must vanish on the set $\{\overline{\theta} = 0\}$. Then, using the equivalence between the primal and the dual forms (3.6) and (4.3) of $\mathcal{C}^{lim}(\overline{\theta})$, we obtain:

$$\begin{aligned} 0 &= \int_Q \Big\{ (\overline{\sigma}_1, \overline{\sigma}_2)\cdot(e_{13}(\overline{v}), e_{23}(\overline{v})) - \overline{\theta}\,\bar{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - \overline{\theta}^{\,-1}\,\bar{j}^{\,*}(\overline{\sigma}_1, \overline{\sigma}_2, 0) \Big\}\, dx \\ &= \int_{Q\cap\{\overline{\theta}>0\}} \Big\{ \overline{\theta}^{\,-1}(\overline{\sigma}_1, \overline{\sigma}_2)\cdot(e_{13}(\overline{v}), e_{23}(\overline{v})) - \bar{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - \bar{j}^{\,*}(\overline{\theta}^{\,-1}(\overline{\sigma}_1, \overline{\sigma}_2, 0)) \Big\}\, \overline{\theta}\, dx \ , \end{aligned}$$

which yields (4.7) thanks to the Fenchel inequality.

Similarly, again using (4.10), the equivalence between (4.1) and (4.4) implies:

$$\int_Q \Big\{ (\overline{\sigma}_1, \overline{\sigma}_2) \cdot (e_{13}(\overline{v}), e_{23}(\overline{v})) - [\overline{j} - k]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - [\overline{j} - k]_+^* (\overline{\sigma}_1, \overline{\sigma}_2, 0) \Big\}\, dx = 0 \ ,$$

which yields (4.8) thanks to the Fenchel inequality.
Finally, the equivalence between (3.5) and (4.1) implies:

$$\int_Q \Big\{ \big(\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - k\big)\, \overline{\theta} - \big[\overline{j} - k\big]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), 0) \Big\}\, dx = 0 \ ,$$

which yields (4.9) since the integrand is nonpositive.

Viceversa, assume that $(\overline{\theta}, \overline{v}, \overline{\sigma})$ satisfy the optimality conditions (4.6)-(4.7)-(4.8)-(4.9).
By (4.6), $\overline{\sigma}$ is admissible for $\mathcal{C}^{lim}(\overline{\theta})$ in its dual form (4.3). Hence,

$$\begin{aligned}
&\langle G, \overline{v} \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0)\, \overline{\theta}\, dx \\
&\leq \sup \Big\{ \langle G, v \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v), e_{23}(v), 0)\, \overline{\theta}\, dx \ : \ v \in TW(Q) \Big\} = \mathcal{C}^{lim}(\overline{\theta}) \\
&= \inf \Big\{ \int_Q \overline{\theta}^{\,-1} \overline{j}^{\,*}(\sigma_1, \sigma_2, 0)\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^2) \ , \ \mathrm{div}_{x'} \sigma = -2\, G_3 \ , \ [[x_1 \sigma_2 - x_2 \sigma_1]] = -2 M_G \Big\} \\
&\leq \int_Q \overline{\theta}^{\,-1} \overline{j}^{\,*}(\overline{\sigma}_1, \overline{\sigma}_2, 0)\, dx \ .
\end{aligned}$$

Using (4.10) one sees that, thanks to (4.7), the first and the last term in the above chain of inequalities agree. Hence $\overline{v}$ and $\overline{\sigma}$ are optimal respectively for the primal and the dual forms (3.6) and (4.3) of $\mathcal{C}^{lim}(\overline{\theta})$.
Similarly, by (4.6), $\overline{\sigma}$ is admissible also for problem (4.4). Hence,

$$\begin{aligned}
&\langle G, \overline{v} \rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j} - k\big]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), 0)\, dx \\
&\leq \sup \Big\{ \langle G, v \rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j}(e_{13}(v), e_{23}(v), 0) - k\big]_+ dx \ : \ v \in TW(Q) \Big\} = \phi(k) \\
&= \inf \Big\{ \int_Q \varphi_k(|\sigma|)\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^2) \ , \ \mathrm{div}_{x'} \sigma = -2\, G_3 \ , \ [[x_1 \sigma_2 - x_2 \sigma_1]] = -2 M_G \Big\} \\
&\leq \int_Q \varphi_k(|\overline{\sigma}|)\, dx = \int_Q \big[\overline{j} - k\big]_+^* (\overline{\sigma}_1, \overline{\sigma}_2, 0)\, dx \ .
\end{aligned}$$

Using (4.10) one sees that, thanks to (4.8), the first and the last term in the above chain of inequalities agree. Hence $\overline{v}$ and $\overline{\sigma}$ are optimal respectively for problems (4.1) and (4.4).
It remains to check that $\overline{\theta}$ is optimal for problem (3.5). Indeed we have

$$\begin{aligned}
&\mathcal{C}^{lim}(\overline{\theta}) + k \int_Q \overline{\theta}\, dx = \langle G, \overline{v} \rangle_{\mathbb{R}^3} - \int_Q \big(\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) - k\big)\, \overline{\theta}\, dx \\
&= \langle G, \overline{v} \rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j} - k\big]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), 0)\, dx = \phi(k) \ ,
\end{aligned}$$

where in the first equality we have used the already proved optimality of $\overline{v}$ for the primal form (3.6) of $\mathcal{C}^{lim}(\overline{\theta})$, in the second equality the optimality condition (4.9), and finally in the third equality the already proved optimality of $\overline{v}$ for problem (4.1). □

Remark 4.6 It is interesting to ask whether, via the optimality system, it is possible to establish that problem (3.5) admits a classical solution (namely, an optimal density with values into $\{0,1\}$). If $(\overline{\theta}, \overline{v}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times L^2(Q;\mathbb{R}^2)$ is an optimal triple, the optimality condition (4.9) reveals that $\overline{\theta}$ is a characteristic function provided the level set $\{\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), 0) = k\}$ (or equivalently the set where $\varphi_k(|\overline{\sigma}|) = \sqrt{\frac{k}{2\eta}}|\overline{\sigma}|)$ has zero Lebesgue measure. Looking at problem (4.4), in the case where $G_3 = 0$, one sees that $\overline{\sigma}$ is optimal if and only if $\overline{\sigma}(\cdot, x_3)$ solves for a.e. x_3 the following section problem for $t = M_G(x_3)$:

$$\alpha_k(t) := \inf\left\{ \int_D \varphi_k(|\sigma|)\, dx' \ : \ \sigma \in L^2(D;\mathbb{R}^2) \ , \ \operatorname{div}\sigma = 0, \ [[x_1\sigma_2 - x_2\sigma_1]] = -2t \right\} .$$

Writing any admissible σ as a rotated gradient, and noticing that $\alpha_k(t) = k\alpha_1\big(\frac{t}{\sqrt{k}}\big)$, one is led to set $s := \frac{t}{\sqrt{k}}$ and to study the solutions $\overline{u}$ of the following minimization problem

$$\inf\left\{ \int_D \varphi_1(|\nabla u|)\, dx' \ : \ u \in H^1(D) \ , \int_D u\, dx' = s \right\} .$$

The homogenization region corresponds then to the set $\{0 < |\nabla \overline{u}| < \sqrt{8\eta}\}$, where the integrand φ_1 is not strictly convex. Does it exist a solution $\overline{u}$ for which this set Lebesgue negligible? So far, this is an open question which deserves in our opinion further investigation. We point out that, for a very similar problem, when D is a square, some numerical experiments seem to predict the existence of a homogenization region of nonzero measure [20]. On the other hand, when D is a disk, it can be shown that the solution $\overline{u}$ is unique and no homogenization region appears [1].

5 The small volume fraction limit

In this section, we investigate the behaviour of optimal configurations when the total amount of material becomes infinitesimal. We will be led to the following conclusion: *for small filling ratios and under the action of a horizontal torsion load, the material distribution in an optimal thin rod with convex section tends to concentrate, section by section, near the boundary of its Cheeger set.*
Let us recall that, under the assumption that D is convex, its *Cheeger set* is the unique solution to the problem

$$\inf_{E \subset \overline{D},\, \mathbb{1}_E \in BV(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla \mathbb{1}_E|}{|E|} \tag{5.1}$$

(see for instance [13, 17, 19]).
As said in the Introduction, optimal configurations for small volume fractions can be deduced from the solutions of problem $\phi(k)$ for large values of k. Hence, in order to prove the afore mentioned concentration phenomenon, we are going to proceed along the following line. We first study the variational convergence, as $k \to +\infty$, of problems $\phi(k)$ suitably rescaled (see Theorem 5.2). Their limit takes the form of a minimization problem over the class of positive measures on Q. The optimal measures, namely the limit of optimal density distributions for $\phi(k)$, can be characterized through Proposition 5.3. In particular, when the load has no vertical component and D is convex,

the solution turns out to be unique and can be explicitly determined as a measure concentrated section by section on the boundary of the Cheeger set of D (see Theorem 5.4).
Let us begin by extending the limit compliance $\mathcal{C}^{lim}(\theta)$ given by (3.6) to the class $\mathcal{M}^+(Q)$ of positive measures μ on $\mathbb{R}^3$ compactly supported in Q by setting

$$\mathcal{C}^{lim}(\mu) := \sup \left\{ \langle G, v\rangle_{\mathbb{R}^3} - \int_Q \overline{j}\big(e_{13}(v), e_{23}(v), 0\big)\, d\mu \ :\ v \in TW(Q) \cap \mathcal{C}^\infty(Q;\mathbb{R}^3) \right\} . \tag{5.2}$$

We point out that in dual form $\mathcal{C}^{lim}(\mu)$ reads

$$\mathcal{C}^{lim}(\mu) = \inf_{\xi \in L^2_\mu(Q;\mathbb{R}^2)} \left\{ \int_Q \overline{j}^{\,*}(\xi_1, \xi_2, 0)\, d\mu \ :\ \mathrm{div}_{x'}(\xi\mu) = -2\, G_3,\ [[x_1(\xi_2\mu) - x_2(\xi_1\mu)]] = -2M_G \right\} \tag{5.3}$$

(this follows by applying Lemma 4.4 in a similar way as repeatedly done in the previous section).
Using definition (5.2), the limit problem $\phi(k)$ in (3.5) can be rewritten as

$$\begin{aligned} \phi(k) &= \inf \left\{ \mathcal{C}^{lim}(\mu) + k \int d\mu \ :\ \mu = \theta\, dx\ ,\ \theta \in L^\infty(Q;[0,1]) \right\} \\ &= \sqrt{2k}\, \inf \left\{ \mathcal{C}^{lim}(\mu) + \frac{1}{2} \int d\mu \ :\ \mu = \theta\, dx\ ,\ \theta \in L^\infty(Q;[0,\sqrt{2k}]) \right\} , \end{aligned} \tag{5.4}$$

where the second equality is obtained multiplying μ by $\sqrt{2k}$ (for $k > 0$).
One is thus led to introduce the following minimization problem on $\mathcal{M}^+(Q)$, as a natural candidate to be the limit problem of $\frac{\phi(k)}{\sqrt{2k}}$ as $k \to +\infty$:

$$\overline{m} := \inf \left\{ \mathcal{C}^{lim}(\mu) + \frac{1}{2} \int d\mu \ :\ \mu \in \mathcal{M}^+(Q) \right\} . \tag{5.5}$$

In the next proposition, we give a useful reformulation of $\overline{m}$ as a maximization problem for a linear form under constraint, which in turn admits a pretty tractable dual form.

Proposition 5.1 *Any optimal measure $\overline{\mu}$ in* (5.5) *satisfies*

$$\mathcal{C}^{lim}(\overline{\mu}) = \frac{1}{2} \int d\overline{\mu} = \frac{\overline{m}}{2} , \tag{5.6}$$

and $\overline{m}$ agrees with the following supremum:

$$\sup_{v \in TW(Q)} \left\{ \langle G, v\rangle_{\mathbb{R}^3} \ :\ \big\|(e_{13}(v), e_{23}(v))\big\|_{L^\infty(Q;\mathbb{R}^2)} \le \frac{1}{2\sqrt{\eta}} \right\} , \tag{5.7}$$

or alternatively with the minimum of the dual problem

$$\min \left\{ \int |\sigma| \ :\ \sigma \in \mathcal{M}(Q;\mathbb{R}^2)\ ,\ \mathrm{div}_{x'}\, \sigma = -\frac{1}{\sqrt{\eta}}\, G_3\ ,\ [[x_1\sigma_2 - x_2\sigma_1]] = -\frac{1}{\sqrt{\eta}}\, M_G \right\} . \tag{5.8}$$

PROOF. Let m_0 denote the supremum in (5.7). For every $t \in \mathbb{R}^+$, by the definition of $\mathcal{C}^{lim}(\mu)$ and the same inf-sup commutation argument already used in the proof of Theorem 4.1, we infer:

$$\begin{aligned}\inf\Big\{ \mathcal{C}^{lim}(\mu) \ :\ \textstyle\int d\mu \le t\Big\} &= \sup_v \inf \Big\{\langle G, v\rangle_{\mathbb{R}^3} - 2\eta \int_Q \big|(e_{13}(v), e_{23}(v))\big|^2 \, d\mu \ :\ \int d\mu \le t\Big\} \\ &= \sup_v \Big\{\langle G, v\rangle_{\mathbb{R}^2} - 2\eta t \big\|(e_{13}(v), e_{23}(v))\big\|^2_{L^\infty(Q;\mathbb{R}^2)}\Big\} = \frac{m_0^2}{2t} \ ,\end{aligned}$$

where the last equality follows by writing $v = s v_0$, with $s \in \mathbb{R}$ and v_0 admissible for problem (5.7), and optimizing in the real variable s.
Then, since by the definition (5.5) of $\overline{m}$ we have

$$\overline{m} = \inf_{t\in\mathbb{R}^+} \Big\{ \mathcal{C}^{lim}(\mu) + \frac{t}{2} \ :\ \int d\mu \le t\Big\} = \inf_{t\in\mathbb{R}^+} \Big(\frac{m_0^2}{2t} + \frac{t}{2}\Big) \ ,$$

and since the function $t \mapsto \Big(\frac{m_0^2}{2t} + \frac{t}{2}\Big)$ attains its minimum on $\mathbb{R}^+$ at $t = m_0$, we deduce that the equality $\overline{m} = m_0$ holds and that any optimal measure $\overline{\mu}$ satisfies (5.6).

The dual form (5.8) of problem (5.7) follows from Lemma 4.4, applied with $X := TW(Q) \cap \mathcal{C}_0(Q;\mathbb{R}^3)$, $Y := \mathcal{C}_0(Q;\mathbb{R}^2)$, $A(v) := (e_{13}(v), e_{23}(v))$, $\Phi(v) := -\langle G, v\rangle_{\mathbb{R}^3}$, and $\Psi(y) = 0$ iff $\|y\|_\infty \le 1/(2\sqrt{\eta})$ (and $+\infty$ otherwise).

□

We are now ready to establish that, as expected, $\overline{m}$ is the limit problem of $\frac{\phi(k)}{\sqrt{2k}}$ as $k \to +\infty$. Actually Theorem 5.2 below shows that such convergence holds true in the variational sense, namely not only for the values of the infima, but also for the corresponding solutions.

Theorem 5.2 *(i) For $k > 0$, the map $k \mapsto \frac{\phi(k)}{\sqrt{2k}}$ is nonincreasing and, as $k \to +\infty$, it converges decreasingly to $\overline{m}$.*

(ii) if θ_k is a solution to the density formulation (3.5) *of $\phi(k)$, up to subsequences θ_k converges weakly * in $L^\infty(Q;[0,1])$ to a solution $\overline{\mu}$ of problem* (5.5).

PROOF. The second equality in (5.4) shows that the map $k \mapsto \frac{\phi(k)}{\sqrt{2k}}$ is nonincreasing and satisfies the inequality $\frac{\phi(k)}{\sqrt{2k}} \ge \overline{m}$. In order to show that it converges to $\overline{m}$ as $k \to +\infty$, we exploit the formulation of $\phi(k)$ given in Theorem 4.1, in which we insert the change of variable $\tilde{v} = v/\sqrt{2k}$. We obtain

$$\frac{\phi(k)}{\sqrt{2k}} = \sup_{v\in TW(Q)} \Big\{\langle G, v\rangle_{\mathbb{R}^3} - \sqrt{2k} \int_Q \big[\overline{j}\big(e_{13}(v), e_{23}(v), 0\big) - \frac{1}{2}\big]_+ \, dx\Big\} \ .$$

Let $v_k = \big(c_k(x_3)(-x_2, x_1), w_k(x)\big)$ be fields in $TW(Q) \cap \mathcal{C}^\infty(Q;\mathbb{R}^3)$ such that

$$\limsup_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k\to+\infty} \Big\{\langle G, v_k\rangle_{\mathbb{R}^3} - \sqrt{2k} \int_Q \big[\overline{j}\big(e_{13}(v_k), e_{23}(v_k), 0\big) - \frac{1}{2}\big]_+ \, dx\Big\} \ .$$

By using the coercivity of $[\overline{j}(z) - k]_+$, the inequality $\phi(k) \geq 0$, and the assumption that G is an admissible load, we may find positive constants C_1 and C_2 such that

$$\begin{aligned}\|(e_{13}(v_k), e_{23}(v_k))\|^2_{L^2(Q;\mathbb{R}^2)} &\leq C_1\sqrt{2k}\int_Q \big[\overline{j}\big(e_{13}(v_k), e_{23}(v_k), 0\big) - \frac{1}{2}\big]_+ \, dx \\ &\leq C_1\langle G, v_k\rangle_{\mathbb{R}^3} \leq C_2\|(e_{13}(v_k), e_{23}(v_k))\|_{L^2(Q;\mathbb{R}^2)} \,.\end{aligned}$$

We deduce that $(e_{13}(v_k), e_{23}(v_k))$ is bounded in $L^2(Q;\mathbb{R}^2)$. Hence there exists a positive constant C such that

$$\begin{aligned} C &\geq \int_Q |c_k'(x_3)(-x_2, x_1) + \nabla_{x'} w_k|^2 \, dx \\ &\geq \inf\Big\{\int_D |(-x_2, x_1) + \nabla w|^2 \, dx' \ : \ w \in H^1(D)\Big\} \cdot \int_I |c_k'(x_3)|^2 \, dx_3 \\ &= \gamma \int_I |c_k'(x_3)|^2 \, dx_3 \ , \end{aligned}$$

where the last equality follows from (3.15) in Lemma 3.5. Applying the Poincaré-Wirtinger inequality, we obtain that c_k is uniformly bounded in $H^1_m(I)$.
By difference, it is also clear that $\nabla_{x'} w_k$ is uniformly bounded in $L^2(Q;\mathbb{R}^2)$, hence w_k is uniformly bounded in $L^2(I; H^1_m(D))$.
Let c and w be the weak limits of c_k and w_k in $H^1_m(I)$ and $L^2(I; H^1_m(D))$ respectively, and set $v := (-c(x_3)x_2, c(x_3)x_1, w)$. Then $v \in TW(Q)$ and $\lim_k e_{\alpha 3}(v_k) = e_{\alpha 3}(v)$ weakly in $L^2(Q)$. Therefore

$$\int_Q [\overline{j}(e_{\alpha 3}(v), 0) - 1/2]_+ \, dx \leq \liminf_k \int_Q [\overline{j}(e_{\alpha 3}(v_k), 0) - 1/2]_+ \, dx = 0 \,.$$

Hence

$$\big\|(e_{13}(v), e_{23}(v))\|_{L^\infty(Q;\mathbb{R}^2)} \leq \frac{1}{2\sqrt{\eta}} \,,$$

that is v is admissible in the definition (5.7) of m_0.
We conclude that

$$\lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} \ \leq \ \lim_{k\to+\infty} \langle G, v_k\rangle_{\mathbb{R}^3} \ = \langle G, v\rangle_{\mathbb{R}^3} \ \leq \ m_0 = \overline{m} \ .$$

(ii) If θ_k is an optimal density for $\phi(k)$, setting $\mu_k := \sqrt{2k}\, \theta_k \, dx$ one has

$$\frac{\phi(k)}{\sqrt{2k}} = \mathcal{C}^{lim}(\mu_k) + \frac{1}{2}\int d\mu_k \ .$$

Since $\mathcal{C}^{lim}(\mu_k) \geq 0$ and since by monotonicity $\frac{\phi(k)}{\sqrt{2k}} \leq \phi(1)$, the above equation implies that the integral $\int d\mu_k$ remains uniformly bounded. Then up to a subsequence there exists $\overline{\mu}$ such that $\mu_k \overset{*}{\rightharpoonup} \overline{\mu}$. By using item (i) already proved, the weak * semicontinuity of the map $\mu \mapsto \mathcal{C}^{lim}(\mu)$, and the definition (5.5) of $\overline{m}$, we obtain

$$\overline{m} = \lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k\to+\infty} \Big\{\mathcal{C}^{lim}(\mu_k) + \frac{1}{2}\int d\mu_k\Big\} \geq \mathcal{C}^{lim}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \geq \overline{m} \ .$$

Hence $\overline{\mu}$ is a solution to problem (5.5).

$\square$

By the convergence statement (ii) in Theorem 5.2, in order to understand which kind of concentration phenomenon occurs for small amounts of material, one needs to answer the following question: what can be said about solutions $\overline{\mu}$ to problem (5.5)? In this direction, let us first show that optimal measures $\overline{\mu}$ are strictly related to solutions $\overline{\sigma}$ to the dual problem (5.8). More precisely, we have:

Proposition 5.3 *If $\overline{\sigma}$ is optimal for problem* (5.8)*, then $\overline{\mu} := |\overline{\sigma}|$ is optimal for problem* (5.5)*. Conversely, if $\overline{\mu}$ is optimal for problem* (5.5)*, and $\overline{\xi}$ is optimal for the dual form* (5.3) *of $\mathcal{C}^{lim}(\overline{\mu})$, then $|\overline{\xi}| = 2\sqrt{\eta}$ $\overline{\mu}$-a.e., and $\overline{\sigma} := \frac{\overline{\xi}}{2\sqrt{\eta}}\,\overline{\mu}$ is optimal for problem* (5.8).

PROOF. Let $\overline{\sigma}$ be optimal for the dual problem (5.8), and set $\overline{\mu} := |\overline{\sigma}|$. Then we have $\left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right| = 1$ $\overline{\mu}$-a.e. and $\int d\overline{\mu} = \overline{m}$. Moreover, since $\overline{\sigma}$ is admissible in (5.8), it holds $\mathrm{div}\,\big(2\sqrt{\eta}E_0(\overline{\sigma}_1, \overline{\sigma}_2, 0)\big) + G \in TW(Q)^{\perp}$, namely

$$\langle G, v\rangle_{\mathbb{R}^3} = 2\sqrt{\eta}\langle(\overline{\sigma}_1, \overline{\sigma}_2), (e_{13}(v), e_{23}(v))\rangle_{\mathbb{R}^3} \qquad \forall v \in TW(Q) \cap \mathcal{C}^{\infty}(Q; \mathbb{R}^3)\ . \tag{5.9}$$

By (5.2), (5.9), the Fenchel inequality and the identity

$$\overline{j}^{\,*}(\xi_1, \xi_2, 0) = \frac{1}{8\eta}|\xi|^2 \qquad \forall\, \xi = (\xi_1, \xi_2, 0)\ ,$$

we get

$$\begin{aligned}\mathcal{C}^{lim}(\overline{\mu}) &= \sup_v \Big\{2\sqrt{\eta}\langle(\overline{\sigma}_1, \overline{\sigma}_2), (e_{13}(v), e_{23}(v))\rangle_{\mathbb{R}^3} - 2\eta\int_Q \big|(e_{13}(v), e_{23}(v))\big|^2\, d\overline{\mu}\Big\} \\ &\leq \frac{1}{8\eta}\int 4\eta\left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right|^2 d\overline{\mu} = \frac{\overline{m}}{2}\ ,\end{aligned}$$

and hence

$$\mathcal{C}^{lim}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \leq \overline{m}\ .$$

Conversely, assume that $\overline{\mu}$ is optimal for problem (5.5), and let $\overline{\xi}$ be optimal for the dual form (5.3) of $\mathcal{C}^{lim}(\overline{\mu})$, that is

$$\int_Q \overline{j}^{\,*}(\overline{\xi}_1, \overline{\xi}_2, 0)\, d\overline{\mu} = \mathcal{C}^{lim}(\overline{\mu})\ . \tag{5.10}$$

Set $\overline{\sigma} := \frac{\overline{\xi}}{2\sqrt{\eta}}\,\overline{\mu}$, and notice that it is admissible for problem (5.8). If we prove that

$$|\overline{\xi}| \leq 2\sqrt{\eta} \quad \overline{\mu}\text{-a.e.}\ , \tag{5.11}$$

we are done: indeed in this case $\overline{\sigma}$ is optimal for problem (5.8) because

$$\int |\overline{\sigma}| = \int \frac{|\overline{\xi}|}{2\sqrt{\eta}}\, d\overline{\mu} \leq \int d\overline{\mu} = \overline{m}\ ,$$

where in the last equality we have applied (5.6).

Let us prove (5.11). By (5.10), if $v_k \in TW(Q)$ is a minimizing sequence for $\mathcal{C}^{lim}(\overline{\mu})$, one has

$$\int_Q \overline{j}^*(\overline{\xi}_1, \overline{\xi}_2, 0)\, d\overline{\mu} = \mathcal{C}^{lim}(\overline{\mu}) = \lim_k \left\{ \langle G, v_k \rangle_{\mathbb{R}^3} - \int_Q \overline{j}\big(e_{13}(v_k), e_{23}(v_k), 0\big)\, d\overline{\mu} \right\} . \tag{5.12}$$

For every k, by (5.9) and since $\overline{\sigma} = \frac{\overline{\xi}}{2\sqrt{\eta}}\overline{\mu}$, it holds

$$\langle G, v_k \rangle_{\mathbb{R}^3} = \int_Q \big[\overline{\xi}_1 e_{13}(v_k) + \overline{\xi}_2 e_{23}(v_k)\big]\, d\overline{\mu} . \tag{5.13}$$

Now, by arguing in a similar way as in the proof of Proposition 5.1 (see also [7, Corollary 2.4]), we observe that the minimizing sequence v_k can be chosen so that $\big|\big(e_{13}(v_k), e_{23}(v_k)\big)\big| \le \frac{1}{2\sqrt{\eta}}$ on Q. Denote by (χ_1, χ_2) a cluster point of $\big(e_{13}(v_k), e_{23}(v_k)\big)$ in $L^2_{\overline{\mu}}(Q; \mathbb{R}^2)$. Then we have

$$\big|(\chi_1, \chi_2)\big| \le \frac{1}{2\sqrt{\eta}} \qquad \overline{\mu}\text{-a.e.} \tag{5.14}$$

and

$$\liminf_k \int_Q \overline{j}\big(e_{13}(v_k), e_{23}(v_k), 0\big)\, d\overline{\mu} \ge \int_Q \overline{j}\big(\chi_1, \chi_2, 0\big)\, d\overline{\mu} . \tag{5.15}$$

By (5.12), (5.13) and (5.15), we obtain the following converse Fenchel inequality

$$\int_Q \overline{j}^*(\overline{\xi}_1, \overline{\xi}_2, 0)\, d\overline{\mu} \le \int_Q \big[\overline{\xi}_1 \chi_1 + \overline{\xi}_2 \chi_2\big]\, d\overline{\mu} - \int_Q \overline{j}\big(\chi_1, \chi_2, 0\big)\, d\overline{\mu} .$$

Hence

$$(\overline{\xi}_1, \overline{\xi}_2, 0) = \overline{j}'\big(\chi_1, \chi_2, 0\big) = 4\eta(\chi_1, \chi_2, 0) , \tag{5.16}$$

where the second equality follows by recalling the explicit form (3.4) of $\overline{j}$.
In turn, (5.16) gives (5.11) in view of (5.14).

□

Thanks to Proposition 5.3, in order to determine optimal measures for problem (5.5), one is reduced to study the solutions to the dual problem (5.8). When the applied torsion load has null vertical component, and the cross section D of the rod is a convex set, problem (5.8) turns out to have a unique solution, which brings into play the Cheeger set of D.

Theorem 5.4 *Assume that $G_3 = 0$ and that D is convex. Denote by C the Cheeger set of D. Then the unique solution to problem* (5.8) *is*

$$\overline{\sigma} := \frac{1}{2\sqrt{\eta}} M_G(x_3) \otimes \frac{1}{|C|} \tau_{\partial C}(x') \mathcal{H}^1 \llcorner \partial C , \tag{5.17}$$

and hence the unique solution $\overline{\mu}$ to problem (5.5) *is*

$$\overline{\mu} = \frac{1}{2\sqrt{\eta}} |M_G(x_3)| \otimes \frac{1}{|C|} \mathcal{H}^1 \llcorner \partial C . \tag{5.18}$$

PROOF. We notice that the constraints imposed on the admissible measures σ in the minimization problem (5.8) only involve the behaviour of $\sigma(\cdot, x_3)$ for each fixed $x_3 \in I$. Therefore, solutions can be searched under the form

$$\sigma(x) = \gamma(x_3) \otimes \nu(x') \quad \text{with } \gamma \in \mathcal{M}(I;\mathbb{R}) \text{ and } \nu \in \mathcal{M}(D;\mathbb{R}^2)\ .$$

In terms of γ and ν, the problem is rewritten as

$$\min \Big\{ \int_I |\gamma| \int_D |\nu| \ :\ \operatorname{div} \nu = 0\ ,\ \Big(\int_D x_1\, d\nu_2(x') - x_2\, d\nu_1(x') \Big) \gamma(x_3) = -\frac{1}{\sqrt{\eta}} M_G(x_3) \Big\}\ .$$

Hence, up to constant multiples, the optimal measures $(\overline{\gamma}, \overline{\nu})$ are uniquely determined respectively as

$$\overline{\gamma}(x_3) := \frac{1}{2\sqrt{\eta}} M_G(x_3)\ ,$$

and an optimal measure $\overline{\nu}$ for the following section problem:

$$\min \Big\{ \int |\nu| \ :\ \nu \in \mathcal{M}(D;\mathbb{R}^2)\,,\ \operatorname{div} \nu = 0\,,\ \int_D \big(x_1\, d\nu_2(x') - x_2\, d\nu_1(x') \big) = -2 \Big\}\ .$$

Since D is assumed to be simply connected, we may write any admissible ν as $(-D_2 u, D_1 u)$, for some u in the space $BV_0(D)$ of bounded variation functions which vanish identically outside $\overline{D}$. So that we arrive at problem

$$\min \Big\{ \int |Du| \ :\ u \in BV_0(D)\ ,\ \int_D u = 1 \Big\}\ . \tag{5.19}$$

This is precisely the relaxed formulation of problem (5.1) on D. When D is *convex*, it is known that problem (5.19) admits a unique solution, which is of the form $\overline{u} = |C|^{-1} \mathbb{1}_C$, where C is the Cheeger set of D. Hence, for bars with convex cross section, the unique solution to problem (5.8) is given by (5.17). By Proposition 5.3, it follows that the unique solution $\overline{\mu}$ to (5.5) is given by (5.18). □

6 Appendix

Proof of Lemma 3.1.
Write any $\Omega \subseteq Q_\delta$ as $\Omega = \big\{ (\delta x', x_3) : (x', x_3) \in \omega \big\}$, so that $\omega \subseteq Q$. Then, calling $\tilde{u} \in H^1(Q_\delta;\mathbb{R}^3)$ an admissible displacement in the definition of $\mathcal{C}(\Omega)$, set $\tilde{u}(x) := \big(\delta^{-2} u_\alpha(\delta^{-1} x', x_3)\ ,\ \delta^{-1} u_3(\delta^{-1} x', x_3) \big)$, so that $u \in H^1(Q;\mathbb{R}^3)$. Thanks to the scaling chosen for the load, it holds $\langle G^\delta, \tilde{u} \rangle_{\mathbb{R}^3} = \delta^{-1} \langle G, u \rangle_{\mathbb{R}^3}$. Moreover, via change of variables, one gets $\int_\Omega j(e(\tilde{u}))\, dx = \int_\omega j\big(e^\delta(u)\big)\, dx$. □

Proof of Lemma 3.3.
The assumption (2.4) on the load implies

$$\delta^{-1} \langle G, u^\delta \rangle_{\mathbb{R}^3} = \delta^{-1} \langle \operatorname{div} \Sigma, u^\delta \rangle_{\mathbb{R}^3} = -\delta^{-1} \langle \Sigma, e(u^\delta) \rangle_{\mathbb{R}^3} = -\delta \langle \Sigma_{\alpha\beta}, e^\delta_{\alpha\beta}(u^\delta) \rangle_{\mathbb{R}^3} - 2 \, \langle \Sigma_{\alpha 3}, e^\delta_{\alpha 3}(u^\delta) \rangle_{\mathbb{R}^3}\ .$$

Therefore, the convergence in (3.11) is immediate once we have proved that the L^2-norm of $e^\delta(u^\delta)$ remains bounded. Since by assumption $\Sigma \in L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$, there exists a positive constant C_1 such that

$$\delta^{-1} \langle G, u^\delta \rangle_{\mathbb{R}^3} \le C_1 \| e^\delta(u^\delta) \|_{L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})}\ .$$

On the other hand, since j is coercive and by assumption $\inf_Q \theta > 0$, we may find a positive constant C_2 such that

$$\int_Q j(e^\delta(u^\delta))\theta \, dx \geq C_2 \|e^\delta(u^\delta)\|^2_{L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})} \, .$$

Hence, exploiting also the assumption that the infimum in (3.10) is a finite constant C_3, we obtain

$$C_2\|e^\delta(u^\delta)\|^2_{L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})} \leq \int_Q j(e^\delta(u^\delta))\theta \, dx \leq \delta^{-1}\langle G, u^\delta\rangle_{\mathbb{R}^3} - C_3 \leq C_1\|e^\delta(u^\delta)\|_{L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})} - C_3 \, .$$

Hence $e^\delta(u^\delta)$ remains bounded in $L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})$ as required. □

Proof of Lemma 3.5.
The positivity of ψ_D is a consequence of the maximum principle.
A minimizing sequence ψ_n for the variational problem in (3.14) converges weakly in $H^1_0(D)$ to a function $\overline{\psi} \in H^1_0(D)$ which solves the Euler equation $-\Delta\overline{\psi} = 2\lambda$ in D, for some $\lambda \in \mathbb{R}$. Thus $\overline{\psi} = \lambda\psi_D$, and

$$\int_D |\nabla\overline{\psi}|^2 \, dx' = 2\lambda \int_D \overline{\psi} \, dx' = 2\lambda = 2\Big(\int_D \psi_D \, dx'\Big)^{-1} = 4\gamma^{-1} \, .$$

If $\overline{w}$ is a solution to (3.15), the Euler equation gives

$$\operatorname{div}\big(((-x_2, x_1) + \nabla\overline{w})\mathbb{1}_D\big) = 0 \quad \text{ in } \mathcal{D}'(\mathbb{R}^2) \, .$$

Hence there exists a function $\psi \in H^1(\mathbb{R}^2)$ such that $((-x_2, x_1) + \nabla\overline{w})\mathbb{1}_D = (\partial_2\psi, -\partial_1\psi)$ in $\mathbb{R}^2$ and $\psi = 0$ in $\mathbb{R}^2 \setminus D$. This implies that ψ solves $-\Delta\psi = 2$ in D and vanishes on ∂D, so that $\psi \llcorner D = \psi_D$.
□

Proof of Lemma 3.6.
To prove (3.16), we argue by contradiction: assume there exists a sequence $v_n \in H^1_m(D;\mathbb{R}^2)$, with

$$\int_D |v_n|^2 \, dx' = 1 \ \forall n \, , \qquad \lim_n \int_D |e(v_n)|^2 \, dx' = 0 \, , \qquad \lim_n \int_D (\nabla\psi_D \wedge v_n) \, dx' = 0 \, .$$

By the first two conditions above and the Korn inequality on D, possibly passing to a subsequence, we deduce that v_n converges strongly in $L^2(D;\mathbb{R}^2)$. Its limit $\overline{v}$ is a rigid motion with zero integral mean, hence it is of the form $\overline{v} = \overline{\lambda}(-x_2, x_1)$ for some constant $\overline{\lambda} \in \mathbb{R}$. Then

$$0 = \lim_n \int_D (\nabla\psi_D \wedge v_n) \, dx' = \overline{\lambda} \int_D x' \cdot \nabla\psi_D \, dx' = -2\overline{\lambda} \int_D \psi_D \, dx' \, ,$$

where the last equality follows integrating by parts and recalling that $\psi_D \in H^1_0(D)$. Thus, since $\int_D \psi_D \, dx' \neq 0$, it must be $\overline{\lambda} = 0$. This implies $\overline{v} = 0$, that is $v_n \to 0$ strongly in $L^2(D;\mathbb{R}^2)$, against the assumption $\|v_n\|_{L^2(D;\mathbb{R}^2)} = 1$ for every n.

In order to show (3.17), up to replacing v by

$$v + \frac{\int_D (\nabla\psi_D \wedge v) \, dx'}{2\int_D \psi_D \, dx'}(-x_2, x_1) \, ,$$

it is not restrictive to assume that $\int_D (\nabla\psi_D \wedge v)\, dx' = 0$. Again by contradiction, let $v_n \in H^1_m(D;\mathbb{R}^2)$ be a sequence such that

$$\int_D |\operatorname{curl} v_n|^2\, dx' = 1 \ \forall n\ , \qquad \lim_n \int_D |e(v_n)|^2\, dx' = 0\ , \qquad \int_D (\nabla\psi_D \wedge v_n)\, dx' = 0 \ \forall n\ .$$

By (3.16) and Korn inequality, we infer that v_n converges strongly to 0 in $H^1_m(D;\mathbb{R}^2)$, which implies in particular that $\operatorname{curl} v_n$ converges strongly to 0 in $L^2(D)$, against the assumption $\|\operatorname{curl} v_n\|_{L^2(D)} = 1$ for every n.

□

Proof of Lemma 3.7.
Let us first estimate the integral mean of c^δ defined in (3.19). Exploiting the hypothesis (3.18) and recalling that $\int_D \psi_D(x')\, dx' = \gamma/2$ (see (3.13)), we have:

$$\begin{aligned}\Big|\int_I c^\delta(x_3)\, dx_3\Big|^2 &= \Big|\frac{2}{\gamma}\int_Q \psi_D(x')c^\delta(x_3)\, dx - \frac{1}{\gamma\delta}\int_Q \psi_D(x')\operatorname{curl}_{x'}(u_1^\delta, u_2^\delta)\, dx\Big|^2 \\ &= \frac{4}{\gamma^2}\Big|\int_Q \psi_D(x')\big[c^\delta(x_3) - \frac{1}{2\delta}\operatorname{curl}_{x'}(u_1^\delta, u_2^\delta)\big]\, dx\Big|^2 \\ &\le C\int_Q \big|c^\delta(x_3) - \frac{1}{2\delta}\operatorname{curl}_{x'}(u_1^\delta, u_2^\delta)\big|^2\, dx\ ,\end{aligned} \tag{6.1}$$

where, in the last line, we have applied the Cauchy-Schwartz inequality. In order to estimate the integral (6.1), we now apply (3.17) in Lemma 3.6 to the field $v = u_\alpha^\delta(\cdot, x_3) - [[u_\alpha^\delta]](\cdot, x_3)$ (which belongs to $H^1_m(D;\mathbb{R}^2)$). Since subtracting from u_α^δ its mean $[[u_\alpha^\delta]]$ does not affect the expressions of the functions $c^\delta(x_3)$, $\operatorname{curl}_{x'}(u_1^\delta, u_2^\delta)$ and $e_{\alpha\beta}(u^\delta)$, we obtain

$$\int_Q \big|c^\delta(x_3) - \frac{1}{2\delta}\operatorname{curl}_{x'}(u_1^\delta, u_2^\delta)\big|^2\, dx \le \frac{C}{\delta^2}\int_Q \big|e_{\alpha\beta}(u_1^\delta, u_2^\delta)\big|^2\, dx\,. \tag{6.2}$$

Combining (6.1) and (6.2), thanks to the L^2-boundedness of $e^\delta(u^\delta)$, we conclude

$$\Big|\int_I c^\delta(x_3)\, dx_3\Big|^2 \le C\delta^2\,. \tag{6.3}$$

We now turn to estimate the derivative of c^δ. We have:

$$(c^\delta)'(x_3) = \int_D (\nabla\psi_D \wedge e_{\alpha 3}^\delta(u^\delta))\, dx' - \frac{1}{2\delta}\int_D (\nabla\psi_D \wedge \nabla_{x'} u_3^\delta)\, dx'\ .$$

Now we notice that the second integral vanishes: indeed, since ψ_D is constant on ∂D, integration by parts gives

$$\int_D (\nabla\psi_D \wedge \nabla_{x'} u_3^\delta)\, dx' = \int_{\partial D} u_3^\delta(\nabla_{\partial D}\psi_D)\, ds = 0\ .$$

Therefore

$$(c^\delta)'(x_3) = \int_D (\nabla\psi_D \wedge e_{\alpha 3}^\delta(u^\delta))\, dx'\ .$$

So we obtain the inequality

$$\left|(c^\delta)'(x_3)\right|^2 \leq \int_D |\nabla\psi_{_D}|^2\, dx' \int_D |e^\delta_{\alpha 3}(u^\delta)|^2\, dx' \ ,$$

and, integrating over I,

$$\int_I \left|(c^\delta)'(x_3)\right|^2 dx_3 \leq \int_D |\nabla\psi_{_D}|^2\, dx' \int_Q |e^\delta_{\alpha 3}(u^\delta)|^2\, dx \ . \tag{6.4}$$

Combining (6.3) and (6.4), we conclude that c^δ is bounded in $H^1(I)$.

□

Proof of Lemma 3.9.
Definition (3.36) implies immediately the inequality $j_0 \leq j$ and also the 2-homogeneity of j_0, since j, and hence j^*, are 2-homogeneous.
We now prove the coercivity of j_0: for a fixed $z \in \mathbb{R}^{3\times 3}_{\text{sym}}$, we consider $\xi := \alpha\lambda_1(z)(e_z \otimes e_z)$, where $\lambda_1(z)$ is the largest (in modulus) eigenvalue of z, e_z is a corresponding eigenvector of norm 1 and α is an arbitrary constant. Since the tensor ξ is degenerate, by definition of j_0 it holds

$$j_0(z) \geq \sup_\alpha \left\{\alpha\lambda_1(z) z\cdot(e_z\otimes e_z) - j^*(\alpha\lambda_1(z)e_z\otimes e_z)\right\} \ .$$

Thanks to the 2-homogeneity of j^* , we obtain

$$j_0(z) \geq |\lambda_1(z)|^2 \sup_\alpha\{\alpha - \alpha^2 \sup_{\|e\|=1} j^*(e\otimes e)\} = \frac{|\lambda_1(z)|^2}{4c} \geq \frac{\|z\|^2}{12\,c} \,,$$

where the constant $c := \sup_{\|e\|=1}\{j^*(e\otimes e)\}$ is clearly strictly positive and finite.
We finally prove (3.37). Applying the identity (3.33) to j and to j_0 we infer, for every $y \in \mathbb{R}^3$:

$$\overline{j_0}(y) = \sup\left\{y\cdot\xi - j_0^*(E_0\xi) \ : \ \xi \in \mathbb{R}^3\right\} \ , \quad \overline{j}(y) = \sup\left\{y\cdot\xi - j^*(E_0\xi) \ : \ \xi\in\mathbb{R}^3\right\}$$

(*cf.* (3.32) for the definition of $E_0\xi$). Then (3.37) follows since $j_0^*(E_0\xi) = j^*(E_0\xi)$ for all $\xi \in \mathbb{R}^3$. Actually, j_0^* and j^* agree on the class of degenerated tensors, see [4, Lemma 3.1]. □

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