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Determination of the closure of the set of elasticity functionals

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Abstract

We determine the closure for Mosco-convergence in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ of the set of elasticity functionals. We prove that this closure coincides with the set of all non-negative lower-semicontinuous quadratic functionals which are objective, i.e. which vanish for rigid motions. The result is still valid if we consider only the set of isotropic elasticity functionals which have a prescribed Poisson coefficient. This shows that a very large family of materials can be reached when homogenizing a composite material with highly contrasted rigidity coefficients.

keywords: Homogenization, Mosco-convergence, Γ-convergence, Composite materials.

Contents

1. Introduction

Finding all possible materials which may, a priori, be the limit of a sequence of elastic materials has a great mathematical and practical interest. Indeed, once the form of the limit energy is known it is enough to consider a few typical situations (and to make a few experiments) to identify the parameters which determine this limit. This is generally the way followed by those who make experiments in order to describe the behavior of composite materials. This paper is devoted to the case of linear elasticity in three-dimensional physical space. Then, the considered functionals are non-negative quadratic forms defined on a functional space, namely on the Sobolev space $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ and take the form

$$F(u) := \int_\Omega f(x, \nabla u(x))\, dx. \tag{1.1}$$

Here Ω is the domain (a convex bounded open set of $\mathbb{R}^3$) occupied by the material. The function $f(x,\cdot)$ is a non-negative quadratic form on the space of 3×3-matrices. Rigid motions, in the linear theory, are those functions u which take the form $u(x) := a + \mathrm{w}\wedge x$ for some given vectors a and w of $\mathbb{R}^3$. A physical law (called objectivity) leads one to consider only functions f which vanish for rigid motions. It is well known that this implies that $f(x,\cdot)$ depends only on the strain tensor, i.e. the symmetric part $e(u)$ of the gradient of u ($2e(u) := \nabla u + \nabla^t u$).

The particular case of isotropic elastic materials is of great interest. In that case the energy density f is determined by two scalar quantities α and β (the Lamé coefficients):

$$f(x, \nabla u(x)) = \alpha(x)\|e(u)\|^2 + \beta(x)(tr(e(u)))^2. \tag{1.2}$$

Here $\|e(u)\|$ denotes the Euclidian norm of the matrix $e(u)$, and $tr(e(u))$ its trace.

We consider a set $\mathcal{S}$ of functionals of type (1.1)-(1.2) and we search for the set of all possible limits of sequences contained in this set. In other words we search for the sequential closure of the considered set of functionals. This problem has been extensively studied under assumptions which ensure that the limit functionals are still elasticity functionals. Suitable assumptions can be [17] the existence of positive constants c, C such that, for any F in $\mathcal{S}$, the energy density f satisfies

$$c\|e(u)\|^2 \le f(x, \nabla u(x)) \le C\|e(u)\|^2. \tag{1.3}$$

This is the case, for instance, of a composite material made of two isotropic materials with comparable rigidity. Then f has the form (1.2) with

$$\alpha = \alpha_1 \mathbf{1}_{\Omega^n} + \alpha_2 \mathbf{1}_{\Omega\setminus\Omega^n} \qquad \beta = \beta_1 \mathbf{1}_{\Omega^n} + \beta_2 \mathbf{1}_{\Omega\setminus\Omega^n} \tag{1.4}$$

where Ω^n is the part of the domain occupied by one of the two components. In these cases, the so-called bounds theory [16],[28], [29],[25], [15], [4], [21],

[11], [19], [2] gives estimates for the limit functional. However the precise determination of the closure of the considered set of functionals has not been completed yet.

But in many composite materials, the different components have very different Lamé coefficients. Then assumptions (1.3) and (1.4) are no more suitable and one must consider, for instance, sequences of functionals in which the ratio α_1/α_2 tends to zero or infinity. It is known that, in this case, the limit functional can be very different from the initial ones. Examples have been given in which the limit functional is non-local [5],[6] or in which the limit functional involves some second derivative of u [26]. Many questions arise :

Can any dependence on the second gradient be reached? In particular can we obtain the following functional:

$$F(u) = \int_\Omega \left(\frac{\partial^2 u_1}{\partial x_1^2} \right)^2 dx \,, \tag{1.5}$$

where x_1 and u_1 denote the components of x and u in a same direction? Such an energy would lead to a very unusual behavior of the material (see [1] for a description of this behavior in the one-dimensional case).

Can the limit functional depend on higher derivatives of u? For instance, can we obtain the following functional:

$$F(u) = \int_\Omega \|\nabla\nabla\nabla u\|^2 \, dx \ ? \tag{1.6}$$

Can the limit functional be

$$F(u) = \sum_{i,j,k,l=1}^{3} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) C_{ijkl}(x) \frac{\partial u_j}{\partial x_i}(x) \, dx \,, \tag{1.7}$$

where $C(x) = (C_{ijkl}(x))_{1\leq i,j,k,l\leq 3}$ is any given positive definite fourth order tensor field satisfying the usual symmetries of elasticity tensors? This question has been affirmatively answered by Cherkaev and Milton [20] (See also [19]). For some specific targets C, an numerical answer is provided by Sigmund [27].

A particular example of the preceeding question is whether a sequence of functionals of type (1.1)-(1.2) with positive coefficients β can lead to a limit of the same type but with a negative coefficient β? Using the definition of the Poisson coefficient ν in terms of Lamé coefficients:

$$\nu := \frac{\beta}{2\beta + \alpha} \,, \tag{1.8}$$

this question can be reformulated in a more classical way: can a material with negative Poisson coefficient be the limit of materials with positive

Poisson coefficients ? In particular, under the same restrictions, can we obtain the degenerate functional (when $\nu = -1$):

$$F(u) = \int_{\Omega} \alpha(x) \left(\|e(u)\|^2 - \frac{1}{3}(tr(e(u)))^2 \right) dx \ ? \tag{1.9}$$

A positive answer to this last question has been given in [18] by the explicit and intricate construction of a sequence of suitable composite materials.

Of course, this list of questions is not exhaustive and constructing explicit sequences of functionals in order to answer to all these questions seems beyond reach. In this paper we give a positive answer to all these questions using an indirect method. We prove that any objective, non-negative quadratic functional can be obtained (provided it is lower semi-continuous for the considered topology).

This result must be compared with the scalar case. We recently determined the closure of the set of diffusion functionals [8],[9],[10]. We proved that this closure coincides with the set of all Dirichlet forms [14] which vanish on constant fields. This scalar result can be easily extended to a vectorial case. Indeed, any functional of type (1.1) with

$$f(x, \nabla(u)(x)) := \sum_{i=1}^{3} \alpha_i(x) \|\nabla u_i(x)\|^2 \tag{1.10}$$

is the sum of three independent diffusion functionals acting on the different components of u. The limit of any sequence of such functionals will still be the sum of three diffusion functionals. Then the Deny-Beurling formula [7],[14] provides a representation for this limit as

$$\sum_{k=1}^{3} \Big(\sum_{i,j=1}^{3} \int_{\Omega} \frac{\partial u_k}{\partial x_i}(x) \frac{\partial u_k}{\partial x_j}(x)\, \eta_{ij}^k(dx) + \int_{\Omega\times\Omega} (u_k(x) - u_k(y))^2 \mu^k(dx, dy) + \int_{\Omega} (u_k(x))^2 \nu^k(dx) \Big) , \tag{1.11}$$

in which, for all $k \in \{1, 2, 3\}$, ν^k and μ^k are non-negative Radon measures respectively on Ω and $\Omega \times \Omega$, while η^k is a Radon measure on Ω taking values in the set of non-negative symmetric matrices. Hence, any limit of a sequence of functionals of type (1.1)-(1.10) is the sum of a diffusion term, a non-local two-points interaction and a killing (or strange [24]) term, but, in no way, can there appear functionals like (1.5) or (1.6). In the present paper we are concerned with elasticity functionals (1.1)-(1.2) and our result is very different: the closure we find is much larger than the set of functionals of type (1.11). In our opinion, this phenomenon is due to the particular structure of the kernel of elasticity functionals.

Note that, as the set of elasticity functionals contains only objective functionals, we cannot obtain killing terms (i.e. terms of type $\int_{\Omega}(u_k(x))^2 \nu^k(dx)$) in a limit functional. For lack of space, we do not study here the case of elasticity functionals subject to some Dirichlet boundary condition. In that case, we could prove that the closure coincides with the whole set of all non-negative quadratic functionals (which are lower semi-continuous for the considered topology).

2. Main result

2.1. Convergence of functionals

The domain Ω is a convex bounded open set of $\mathbb{R}^3$. All the functionals we consider in this paper are defined on the Lebesgue space $\mathrm{L}^2(\Omega,\mathbb{R}^3)$. They are proper, convex and lower-semicontinuous in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$. We denote by $\mathfrak{P}$ this set of functionals and we introduce the following notion of convergence which is particularly adapted to our problem:

Definition 1 *We say that a sequence of functionals (F_n) in $\mathfrak{P}$ τ-converges to a functional F, and we write $F_n \stackrel{\tau}{\longrightarrow} F$, if and only if it satisfies the three following properties:*
i) Lower-bound inequality: For any sequence (u_n) converging weakly to some u in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$,

$$\liminf_{n\to\infty} F_n(u_n) \geq F(u) \ . \tag{2.1}$$

ii) First upper-bound inequality: For every u in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$, there exists an approximating sequence (u_n) converging to u strongly in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ such that

$$\limsup_{n\to\infty} F_n(u_n) \leq F(u) \ . \tag{2.2}$$

iii) Second upper-bound inequality: For any u in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$, there exists a sequence (u_n) converging to u for the strong topology of $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ such that

$$\limsup_{n\to\infty} F_n(u_n) \leq F(u) \ . \tag{2.3}$$

Remark 1 *Any density result established for the τ-convergence is also valid for the Mosco-convergence in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ and for the Γ-convergence for the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$.*

The notion of τ-convergence is closely related to the notion of Γ-convergence introduced by De Giorgi [13] which is generally adapted to study the limit of variational problems and to the notion of Mosco-convergence introduced by U. Mosco [22] which is more particularly adapted to convex cases. For a detailed description of these theories we refer to [12] and [3]. Indeed points (i) and (ii) of our definition are equivalent to Mosco-convergence in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ of the sequence (F_n) to F and point (iii) states moreover that the restrictions to $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ of the functionals F_n Mosco-converge in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ to the corresponding restriction of F: roughly speaking, τ-convergence is Mosco-convergence in both $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ and $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. The notion of Γ-convergence is still weaker: to prove the Γ-convergence of (F_n) to F in a given topology, one has to establish the lower-bound and upper-bound inequalities (2.1) and (2.2) when (u_n) tends to u for this topology. It is easy to check that τ-convergence of (F_n) to F implies its Γ-convergence for the strong and the weak topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ and the Γ-convergence of the restrictions to $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ for the strong and the weak topology of $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. Remark 1 is due to the fact that, the stronger is the notion of convergence, the stronger is a density result.

It is classical to shorten proofs of asymptotic problems by considering only sequences with bounded energy. Indeed:

Remark 2 *It is clear that a τ-convergence result is proved if for every subsequence (not relabeled) of (F_n) one considers in (i) only sequences (u_n) with bounded "energy" (i.e. such that $F_n(u_n) < M < +\infty$) and in (ii) and (iii) only functions u such that $F(u) < +\infty$. In the same way it is usual to shorten the proofs of convergence by considering only regular functions u in points (ii) and (iii) of Definition 1: if F is continuous for the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$, both points (ii) and (iii) of Definition 1 will be proved once it is proved that*

iv) for any u in $C^\infty(\Omega,\mathbb{R}^3)$, there exists a sequence (u_n) converging to u for the strong topology of $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ such that

$$\limsup_{n\to\infty} F_n(u_n) \leq F(u) \ . \tag{2.4}$$

The next property will bring to the fore a first interest in the notion of τ-convergence. Indeed this convergence is stable when adding elements of a wide class of perturbations.

Property 1 *Let us denote by $\mathfrak{R}$ the set of all functionals in $\mathfrak{P}$ which are either continuous for the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$, or continuous for the strong topology of $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ with a domain contained in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. We have*

$$F_n \xrightarrow{\tau} F \quad \text{and} \quad G \in \mathfrak{R} \quad \Rightarrow \quad F_n + G \xrightarrow{\tau} F + G. \tag{2.5}$$

Proof: Let (u_n) be a sequence weakly converging to u in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$. As $G \in \mathfrak{P}$ is lower-semicontinuous for the weak topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$, the τ-convergence of (F_n) to F implies the lower-bound inequality:

$$\liminf_{n\to\infty}(F_n + G)(u_n) \geq \liminf_{n\to\infty} F_n(u_n) + \liminf_{n\to\infty} G(u_n) \geq F(u) + G(u) \ ,$$

and point (i) of Definition 1 is proved. Now, let $u \in \mathrm{H}^1(\Omega,\mathbb{R}^3)$. The τ-convergence of (F_n) to F implies the existence of a sequence (u_n) converging strongly to u in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ and satisfying

$$\limsup_{n\to\infty} F_n(u_n) \leq F(u) \ . \tag{2.6}$$

As G, in any case, is continuous for the strong topology of $\mathrm{H}^1(\Omega,\mathbb{R}^3)$, we have also

$$\limsup_{n\to\infty}(F_n+G)(u_n) \le (F+G)(u) \tag{2.7}$$

and point (iii) is proved. Clearly, this proves also point (ii) when the domain of G is contained in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. In the other case, G is continuous for the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$. The τ-convergence of (F_n) to F states the existence of a sequence (u_n) which converges strongly to u in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ and satisfies the upper-bound (2.6). Due to the continuity of G, Inequality (2.7) and then point (ii) still holds. □

Definition 2 *Let $\mathfrak{U}$ be a subset of $\mathfrak{P}$. We call $\overline{\mathfrak{U}}$ the closure of $\mathfrak{U}$ and we define it to be the set of all τ-limits of sequences in $\mathfrak{U}$.*

The next property is essential in our proofs. It brings to the fore a second interest in the notion of τ-convergence.

Property 2 *For any subset $\mathfrak{U}$ of $\mathfrak{P}$, we have $\overline{\overline{\mathfrak{U}}}=\overline{\mathfrak{U}}$.*

Proof: The point is that, like Mosco-convergence and unlike Γ-convergence, τ-convergence is metrizable at least on a large part of $\mathfrak{P}$, namely on the set $\mathfrak{P}_{pr}$ of all functionals in $\mathfrak{P}$ whose domain intersects $\mathrm{H}^1(\Omega,\mathbb{R}^3)$.

It is proved in [3] (section 3.5) that the topology of Mosco-convergence is metrizable on the set of proper lower-semicontinuous convex functionals: there exists a metric d_1 on $\mathfrak{P}$ such that the Mosco-convergence in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ of a sequence (F_n) to a functional F is equivalent to the convergence of $d_1(F_n,F)$ to zero.

Let $F\in\overline{\overline{\mathfrak{U}}}$. By definition, there exist a sequence (F_n) in $\overline{\mathfrak{U}}$ which τ-converges to F as n tends to infinity and, for any n, a sequence (F_m^n) in $\mathfrak{U}$ which τ-converges to F_n as m tends to infinity. As τ-convergence implies Mosco-convergence in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$, as the last convergence is metrizable, then there exists a sequence $(F_{m(n)}^n)$ which Mosco-converges in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ to F as n tends to infinity.

If F belongs to $\mathfrak{P}\setminus\mathfrak{P}_{pr}$, this is enough to state the τ-convergence of (F_n) to F. Indeed $F(u)=+\infty$ for any $u\in\mathrm{H}^1(\Omega,\mathbb{R}^3)$ and owing to Remark 2, point (iii) of Definition 1 is irrelevant.

If F belongs to $\mathfrak{P}_{pr}$, we can also assume, possibly extracting subsequences, that every F_n and F_m^n belongs also to $\mathfrak{P}_{pr}$. Now let us prove that τ-convergence is metrizable on $\mathfrak{P}_{pr}$. The restriction on $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ of any element of $\mathfrak{P}_{pr}$ belongs to the set of proper, convex functionals on $\mathrm{H}^1(\Omega,\mathbb{R}^3)$ which are lower-semicontinuous for the topology of $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. Then, there exists a metric d_2 on this set corresponding to Mosco-convergence in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. Let us introduce the distance $d:=d_1+d_2$ on $\mathfrak{P}_{pr}$. The convergence of d to zero is equivalent to τ-convergence. There exists a sequence $(F_{m(n)}^n)$ which τ-converges in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ to F as n tends to infinity.

Hence, in both cases, F belongs to $\overline{\mathfrak{U}}$. □

Property 3 *Let $\mathfrak{U}$ be a convex cone contained in $\mathfrak{R}$. Let F and G be in $\overline{\mathfrak{U}}$ and assume that F belongs to $\mathfrak{R}$. Then $F+G$ belongs to $\overline{\mathfrak{U}}$.*

Indeed, there exist sequences (F_n) and (G_n) in $\mathfrak{U}$ which τ-converge respectively to F and G. Then Property 1 states that $F+G_n$ τ-converges to $F+G$ and that, for all n, F_m+G_n τ-converges to $F+G_n$ as m tends to infinity. Hence $F+G$ belongs to $\overline{\overline{\mathfrak{U}}}$ and the result is a consequence of Property 2.

Property 4 *Let (F_n) be a non-decreasing sequence of functionals in $\mathfrak{P}$ which pointwise converges to a functional F. Then (F_n) τ-converges to F.*

Proof: We have, for any $n > n_0$, and any $v \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$,

$$F(v) \geq F_n(v) \geq F_{n_0}(v). \tag{2.8}$$

Let (u_n) be a sequence which converges weakly in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ to some u. For any $n_0 \in \mathbb{N}$, the functional F_{n_0} is lower-semicontinuous. Then inequalities (2.8) give

$$\liminf_n F_n(u_n) \geq \liminf_n F_{n_0}(u_n) \geq F_{n_0}(u)\ .$$

Passing to the limit when n_0 tends to infinity in the previous inequality, leads to the lower-bound inequality: $\liminf_n F_n(u_n) \geq F(u)$. The upper-bound inequalities (points (ii) and (iii) of Definition 1) are obviously assured by the pointwise convergence of (F_n) to F. Indeed, it is enough to choose as approximating sequence, the constant one $u_n := u$. □

2.2. Quadratic forms

Objective quadratic forms: Let us introduce the set $\mathcal{R} \subset \mathrm{L}^2(\Omega, \mathbb{R}^3)$ of *rigid motions*: that is the set of functions $u \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$ for which there exist a and b in $\mathbb{R}^3$ such that, for almost every $x \in \Omega$, $u(x) = a + b \wedge x$.

We call *objective* a functional in $\mathfrak{P}$ which vanishes for any rigid motion. We denote by $\mathfrak{Q} \subset \mathfrak{P}$ the set of objective quadratic forms, i.e. the set of all lower-semicontinuous non-negative quadratic functionals F from $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ to $\mathbb{R}^+ \cup \{+\infty\}$ which satisfy the following objectivity property:

$$\forall r \in \mathcal{R},\ F(r) = 0 \tag{2.9}$$

They are characterized, in addition to (2.9), by the fact that, for any u and v in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ and any $t \geq 0$,

$$F(u) \geq 0,\ F(tu) \leq t^2 F(u),\ F(u+v) + F(u-v) \leq 2F(u) + 2F(v). \tag{2.10}$$

Remark 3 *The set of all objective quadratic forms is closed: $\overline{\mathfrak{Q}} = \mathfrak{Q}$.*

Indeed, it is easily checked that any τ-limit is lower-semicontinuous in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ and that properties (2.9) and (2.10) pass to the limit by τ-convergence. Note that this closure result remains true even if one uses Γ-convergence in the strong topology of $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ instead of τ-convergence

[23]. This is useful if one intends to interpret our density results in terms of Γ-convergence for the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$.

Elasticity functionals: The elasticity functionals belong to $\mathfrak{Q}$. Let us fix, from now on, a Poisson coefficient ν in $(-1,1/2)$. We consider the set $\mathfrak{E}_\nu$ of isotropic elasticity functionals whose Poisson coefficient coincides with ν. More precisely, denoting

$$j_\nu(u) := \|e(u)\|^2 + \frac{\nu}{(1-2\nu)} tr(e(u))^2 \ , \tag{2.11}$$

and

$$E_{\alpha,\nu}(u) := \begin{cases} \displaystyle\int_\Omega \alpha(x) j_\nu(u)\,dx, & \text{if } u \in \mathrm{H}^1(\Omega,\mathbb{R}^3), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.12}$$

the set $\mathfrak{E}_\nu$ is

$$\mathfrak{E}_\nu := \left\{ E_{\alpha,\nu} \ ; \alpha \in \mathrm{L}^\infty(\Omega,\mathbb{R}^+),\ \alpha^{-1} \in \mathrm{L}^\infty(\Omega,\mathbb{R}^+) \right\} \ . \tag{2.13}$$

The aim of this paper is to determine the closure $\overline{\mathfrak{E}}_\nu$ of $\mathfrak{E}_\nu$.

It is worth noticing that simple algebraic comparisons between quadratic forms on the set of symmetric matrices lead to

Remark 4 *For any u, we have*

$$(1+\nu)(e_{33}(u))^2 \le j_\nu(u) \tag{2.14}$$

and

$$k_\nu \|e(u)\|^2 \le j_\nu(u) \le K_\nu \|e(u)\|^2, \tag{2.15}$$

where k_ν and K_ν are the positive constants: $k_\nu := \min(1, \frac{1+\nu}{1-2\nu})$, $K_\nu := \max(1, \frac{1+\nu}{1-2\nu})$.

Continuous forms: The subset of $\mathfrak{Q}$ of functionals which are continuous with respect to the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ will play an important role. We denote it by $\mathfrak{C}$.

Remark 5 *The set $\overline{\mathfrak{E}}_\nu \cap \mathfrak{C}$ is a convex cone.*

Indeed, $\mathfrak{E}_\nu$ and $\mathfrak{C}$ are convex cones included in $\mathfrak{R}$. If F and G belong to $\overline{\mathfrak{E}}_\nu \cap \mathfrak{C}$, Property 3 ensures that $F+G \in \overline{\mathfrak{E}}_\nu$.

Discrete forms: For any function $g > 0$ in $\mathrm{L}^\infty(\mathbb{R}^3,\mathbb{R}^+)$, any family $(x_i) \in \Omega^p$ of p distinct points in Ω and any non-negative quadratic form q from $(\mathbb{R}^3)^p$ to $\mathbb{R}^+$, we denote $D_{q,(x_i),g}$ the quadratic form defined, for any $u \in \mathrm{L}^2(\Omega,\mathbb{R}^3)$, by

$$D_{q,(x_i),g}(u) := \int_{\tilde{\Omega}} q(u(x_1+x), u(x_2+x), \ldots, u(x_p+x)) g(x)\,dx, \tag{2.16}$$

where $\widetilde{\Omega}$ denotes $\widetilde{\Omega} := \cap_{i=1}^{p}(\Omega - x_i)$. It is easy to verify that such a functional is continuous with respect to the strong topology of $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ and that it is objective if and only if, for all $v \in \mathbb{R}^3$,

$$q(v, v, \ldots, v) = 0 \quad \text{and} \quad q(v \wedge x_1, v \wedge x_2, \ldots, v \wedge x_p) = 0. \tag{2.17}$$

We will refer to Property (2.17) as the *objectivity of* $(q, (x_i))$. The set of all objective forms $D_{q,(x_i),g}$ is denoted $\mathfrak{Q}_p$.

Rank one discrete forms: For any function $g > 0$ in $\mathrm{L}^\infty(\mathbb{R}^3, \mathbb{R}^+)$, any family $(x_i) \in \Omega^p$ of p distinct points in Ω and any family of "forces" $(f_i) \in (\mathbb{R}^3)^p$, we denote by $L_{(f_i),(x_i),g}$ the quadratic form defined, for any $u \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$, by

$$L_{(f_i),(x_i),g}(u) := \int_{\widetilde{\Omega}} \left(\sum_{i=1}^{p} f_i \cdot u(x_i + x) \right)^2 g(x)\, dx. \tag{2.18}$$

The conditions for which such a functional is objective are obvious: the family (x_i, f_i) has to satisfy

$$\sum_{i=1}^{p} f_i = 0 \quad \text{and} \quad \sum_{i=1}^{p} x_i \wedge f_i = 0\ . \tag{2.19}$$

We will refer to this equilibrium property as the *balance of the system of forces* (f_i, x_i). The set of all objective forms $L_{(f_i),(x_i),g}$ is a subset of $\mathfrak{Q}_p$. We denote it $\mathfrak{L}_p$.

Elementary forms: The case $p = 2$ plays a major role in this study. It is easy to verify that $\mathfrak{Q}_2$ coincides with $\mathfrak{L}_2$ which contains only "elementary" functionals of the form:

$$S_{(x_1,x_2),g}(u) = \int_{\widetilde{\Omega}} \mathrm{s}_{(x_1,x_2)}(u(x_1 + x), u(x_2 + x)) g(x)\, dx \tag{2.20}$$

where $\mathrm{s}_{(x_1,x_2)}$ is the quadratic form defined, for any $(u_1, u_2) \in (\mathbb{R}^3)^2$, by

$$\mathrm{s}_{(x_1,x_2)}(u_1, u_2) = \left((u_1 - u_2) \cdot \frac{x_1 - x_2}{\|x_1 - x_2\|} \right)^2 \tag{2.21}$$

Note that, from a mechanical point of view, $\mathrm{s}_{(x_1,x_2)}$ corresponds to a "spring-like" interaction between points x_1 and x_2.

2.3. Main result

This paper is devoted to the proof of the following density result:

Theorem 1 *Let $\nu \in (-1, 1/2)$. The closure of the set $\mathfrak{E}_\nu$ of elasticity functionals coincides with the set of all objective, non-negative and lower-semicontinuous quadratic forms: $\overline{\mathfrak{E}_\nu} = \mathfrak{Q}$.*

As noticed in Remark 3, the fact that $\overline{\mathfrak{E}_\nu} \subset \mathfrak{Q}$ is clear. The difficulty is to prove that any element of $\mathfrak{Q}$ is the limit of some sequence in $\mathfrak{E}_\nu$.

The proof of this result needs several steps. We first prove in section 3 that $\mathfrak{L}_2 \subset \overline{\mathfrak{E}_\nu}$. This means that any two point (spring-like) non-local interaction can be obtained. To this aim, we exhibit an explicit composite material which has the desired effective properties. This step is the longest one and the more technical but the result of it is the foundation on which we base the sequel of the proof.

The second fundamental tool is given by Theorem 4 which allows us to consider indirect interactions. From a mechanical point of view, multipoint interactions through a truss-like union of springs can be obtained. This tool enables us to generalize the first result and to prove successively that $\mathfrak{L}_4$ then, using an induction argument, $\mathfrak{L}_p$ for any $p > 4$ are included in $\overline{\mathfrak{E}_\nu}$. The generalization of the last result to $\mathfrak{Q}_p$ does not present any difficulty.

In a third step (section 5), we approximate any continuous functional by elements of $\mathfrak{Q}_p$. The difficulty of this discretization is due to the necessity of using only objective approximating functionals. All these results are summarized in Theorem 7 which states

$$\mathfrak{C} \subset \overline{\mathfrak{E}_\nu}. \tag{2.22}$$

The last step is standard. We use the well known Moreau-Yosida approximation, to approximate any element of $\mathfrak{Q}$ by a sequence of continuous functionals. Theorem 8 states

$$\mathfrak{Q} \subset \overline{\mathfrak{C}}. \tag{2.23}$$

The proof of Theorem 1 is completed by considering (2.22), (2.23) and Property 2. □

3. Elementary non-local interactions

The goal of this section is to prove that any two point non-local interaction belongs to the closure $\overline{\mathfrak{E}_\nu}$ of the set of isotropic elasticity functionals with Poisson coefficient ν.

Theorem 2 *We have $\mathfrak{L}_2 \subset \overline{\mathfrak{E}_\nu}$.*

Proof: Let us consider $S_{(x_1,x_2),g}$ in $\mathfrak{L}_2$. There exists an increasing sequence of functions g_n with compact support in $\widetilde{\Omega} := (\Omega - x_1) \cap (\Omega - x_2)$ converging to g almost everywhere in $\widetilde{\Omega}$. Clearly, the increasing sequence $S_{(x_1,x_2),g_n}$ pointwise converges to $S_{(x_1,x_2),g}$ and then, due to Property 4, it τ-converges to $S_{(x_1,x_2),g}$. Therefore, owing to Property 2, it is enough to prove that $S_{(x_1,x_2),g_n}$ belongs to $\overline{\mathfrak{E}_\nu}$. Then we can consider, without loss of generality, that g has compact support in $\widetilde{\Omega}$.

Moreover, as $\mathfrak{L}_2 \subset \mathfrak{C} \subset \mathfrak{R}$, Remark 5 allows us to split the function g in many parts and then to assume, without loss of generality, that the support of g is contained in a cube $C := x_0 + (0,d)^3$ which satisfies

$$x_1 + C \subset \Omega, \quad x_2 + C \subset \Omega \quad \text{and} \quad (x_1 + C) \cap (x_2 + C) = \emptyset. \tag{3.1}$$

By applying a translation and a homothety, we may assume that

$$C := (0,1)^3. \tag{3.2}$$

When g satisfies these restrictive assumptions, we complete the proof by constructing explicitly a sequence $E_{\alpha_n,\nu}$ of elasticity functionals and by proving the τ-convergence of this sequence to $S_{(x_1,x_2),g}$. This is the subject of the two following subsections.

3.1. Description of a composite elastic material

Let us define a composite material which leads to the desired effective properties. Let n denote a sequence of integers tending to infinity and (r_n) a sequence of real numbers tending to zero in such a way that

$$\lim_{n\to\infty} n^{-3} |\ln r_n| = +\infty. \tag{3.3}$$

For any n, we divide the cube C into n^3 small cubes

$$C_I^n := \left((\frac{i-1}{n}, \frac{i}{n}) \times (\frac{j-1}{n}, \frac{j}{n}) \times (\frac{k-1}{n}, \frac{k}{n}) \right), \tag{3.4}$$

where $I = (i,j,k)$ belongs to the set $\{1 \cdots n\}^3$ which, in the sequel, we identify with $\mathcal{I}^n := \{1 \cdots n^3\}$.

The non-local interaction is simulated by high rigidity fibers: they are very thin cylinders of axis $x_2 - x_1$ and radius

$$r_I^n := r_n \left(\frac{1}{\sqrt{n}} + \frac{\|x_2 - x_1\| n^3}{\pi(1+\nu)} \int_{C_I^n} g(x)dx \right)^{\frac{1}{2}}. \tag{3.5}$$

We fix the end points of the fibers by introducing a family (z_I^n) in the following way: let c_I^n be the center of C_I^n, let Δ_I^n denote the straight line passing through the point z_I^n and parallel to $x_2 - x_1$, we impose the family (z_I^n) to satisfy

$$\|z_I^n - c_I^n\| < (8n)^{-1}, \ \forall I \in \{1 \cdots n^3\} \tag{3.6}$$
$$d(\Delta_I^n, z_{I'}^n) > 2n^{-2}, \forall I \neq I' \in \{1 \cdots n^3\}. \tag{3.7}$$

The existence of such a family can be easily proved, using an induction argument, by proving that, for any $p \in \{1, \dots, n^3\}$, there exists a family $(z_1^n, \dots, z_p^n)$ satisfying $\mathcal{P}_p$:

$$\mathcal{P}_p \ : \ \begin{cases} \|z_I^n - c_I^n\| < (8n)^{-1}, \forall I \leq p, \\ d(\Delta_I^n, z_{I'}^n) > 2n^{-2}, \forall I \neq I' \leq p. \end{cases}$$

When $p = 1$, it is enough to choose $z_1^n = c_1^n$. Assume the existence of $(z_1^n, z_2^n, \cdots, z_{p-1}^n)$ satisfying $\mathcal{P}_{p-1}$. It is easy to check [1] that, for n large enough, the number of indices $I \leq p-1$ such that the cylinders with axis Δ_I^n and radius $2n^{-2}$ which intersect the ball $B(c_p^n, (8n)^{-1})$ is smaller than n. As the volume of each of these intersections is of order of n^{-5}, the total volume of these intersections is much smaller than the volume of the ball $B(c_p^n, (4n)^{-1})$. We can choose z_p^n in the remaining part and the family $\{z_1^n, z_2^n, \cdots, z_p^n\}$ fulfills $\mathcal{P}_p$.

Let us consider, in C, the balls:

$$\mathcal{B}_I^n := B\left(z_I^n, n^{-2}\right), \tag{3.8}$$

and, in Ω, the cylinders $\mathcal{F}_I^n$:

$$\mathcal{F}_I^n := z_I^n + \{x' + (1-t)x_1 + tx_2 \ ; \ x' \perp (x_2 - x_1), \ \|x'\| < r_I^n, \ t \in (0,1)\}.$$

[1] Let $\mathcal{D}$ be the straight line passing through the point c_p^n and parallel to $x_2 - x_1$, the condition $d(\Delta_I^n, c_p^n) < (8n)^{-1} + 2n^{-2}$ is equivalent to $d(\mathcal{D}, z_I^n) < (8n)^{-1} + 2n^{-2}$ which implies $d(\mathcal{D}, c_I^n) < (4n)^{-1} + 2n^{-2}$ and (for $n > 5$) $d(\mathcal{D}, c_I^n) < (3n)^{-1}$. Let us assume without loss of generality that $\mathcal{D}$ makes with the first vector of the basis an angle smaller than $\pi/4$. We notice that $d(\mathcal{D}, c_{ijk}^n) < (3n)^{-1}$ and $d(\mathcal{D}, c_{ij'k'}^n) < (3n)^{-1}$ imply $d(c_{ijk}^n, c_{ij'k'}^n) < 2\sqrt{2}(3n)^{-1}$. Then $j' = j$ and $k' = k$.

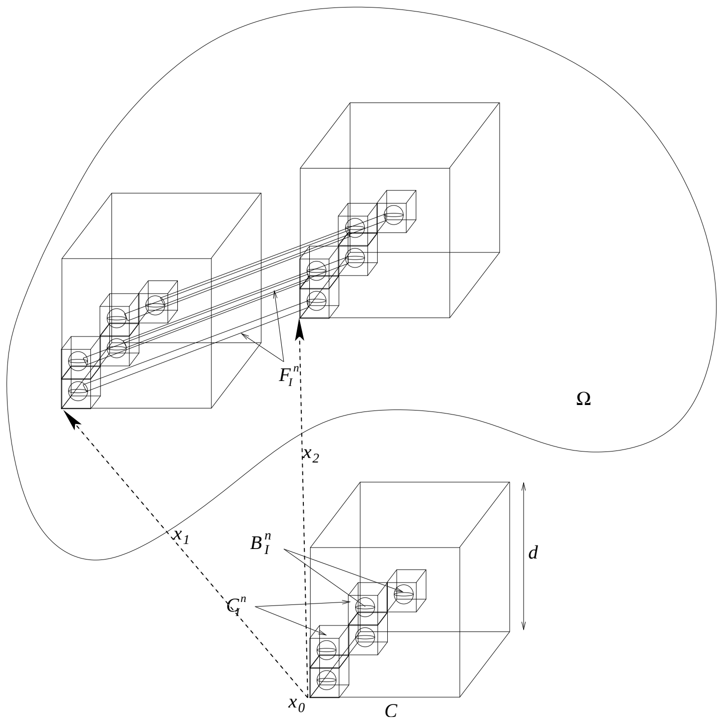

Figure 1: The composite material.

We define the reinforcing part $\Omega^n \subset \Omega$ (cf. Figure 1) by:

$$\Omega^n := \bigcup_{I \in \mathcal{I}^n} (\mathcal{F}_I^n) \cup (x_1 + \mathcal{B}_I^n) \cup (x_2 + \mathcal{B}_I^n)\,. \tag{3.9}$$

Note that the way we chose the family (z_I^n) prevents any collision between the fibers. This was not a priori obvious (cf. Figure 1) since the number n^3 of fibers is very large. Finally, we fix the rigidity of our composite material by setting

$$\alpha_n(x) := \begin{cases} n^{-1/2} & \text{if } x \in \Omega \setminus \Omega^n, \\ r_n^{-2} n^{-3} & \text{if } x \in \Omega^n. \end{cases} \tag{3.10}$$

Theorem 2 will be proved once it is proved that the sequence $(E_{\alpha_n,\nu})$ τ-converges to $S_{(x_1,x_2),g}$. This is the subject of the next subsection.

3.2. A homogenization theorem for elasticity functionals with a non-local limit

This section is devoted to the proof of the following theorem:

Theorem 3 *The sequence of elasticity functionals $(E_{\alpha_n,\nu})$ defined in section 3.1 τ-converges to $S_{(x_1,x_2),g}$.*

3.2.1. Preliminary lemmas The first lemma estimates the energy of an extensional elastic beam. Let $\varepsilon > 0$ and $\ell > 0$ be two real numbers. Let $D(0,\varepsilon) \subset \mathbb{R}^2$ denote the disk of radius ε. In the cylinder $D(0,\varepsilon)\times(0,\ell) \subset \mathbb{R}^3$, we set $x = (x', x_3)$ and $u = (u', u_3)$.

Lemma 1 *For all* $0 < a < b < \ell$ *and* $u \in \mathrm{H}^1(D(0,\varepsilon)\times(0,\ell),\ \mathbb{R}^3)$, *we have*

$$\int_{D(0,\varepsilon)\times(0,\ell)} j_\nu(u)\,dx \geq \frac{\pi\varepsilon^2}{\ell}(1+\nu)\left(⨍_{D(0,\varepsilon)\times(0,a)} u_3 - ⨍_{D(0,\varepsilon)\times(b,\ell)} u_3\right)^2.$$

Proof: In $D(0,\varepsilon)\times(0,\ell)$, we set $x = (x', x_3)$. Owing to Remark 4, we have, for all $0 < y < z < \ell$,

$$\begin{aligned}\int_{D(0,\varepsilon)\times(0,\ell)} j_\nu(u)\,dx &\geq (1+\nu)\int_{D(0,\varepsilon)}\Big(\int_0^\ell \big(\frac{\partial u_3}{\partial x_3}\big)^2\,dx_3\Big)\,dx' \\ &\geq (1+\nu)\int_{D(0,\varepsilon)}\Big(\int_y^z \big(\frac{\partial u_3}{\partial x_3}\big)^2\,dx_3\Big)\,dx' \\ &\geq (1+\nu)\frac{1}{z-y}\int_{D(0,\varepsilon)}\Big(\int_y^z \frac{\partial u_3}{\partial x_3}\,dx_3\Big)^2 dx' \\ &\geq (1+\nu)\frac{1}{\ell}\int_{D(0,\varepsilon)}(u_3(x',z)-u_3(x',y))^2\,dx'.\end{aligned}$$

Taking the mean value of this last term for $y \in (0,a)$ and $z \in (b,\ell)$, Jensen's inequality and Fubini's theorem imply

$$\begin{aligned}&\int_{D(0,\varepsilon)\times(0,\ell)} j_\nu(u)\,dx \\ &\geq \frac{(1+\nu)}{\ell a(\ell-b)}\int_{D(0,\varepsilon)}\Big(\int_0^a\Big(\int_b^\ell (u_3(x',z)-u_3(x',y))^2\,dz\Big)\,dy\Big)\,dx' \\ &\geq \frac{(1+\nu)}{\ell a^2(\ell-b)^2\pi\varepsilon^2}\left(\int_{D(0,\varepsilon)}\Big(a\int_b^\ell u_3(x',z)\,dz - (\ell-b)\int_0^a u_3(x',y)\,dy\Big)\,dx'\right)^2 \\ &\geq \frac{(1+\nu)\pi\varepsilon^2}{\ell}\left(⨍_{D(0,\varepsilon)\times(0,a)} u_3(x)\,dx - ⨍_{D(0,\varepsilon)\times(b,\ell)} u_3(x)\,dx\right)^2. \qquad (3.11)\end{aligned}$$

□

The next Lemma proves that, for a thin cylinder, the previous inequality is optimal. Moreover it estimates the energy of the interaction of such a cylinder with a surrounding similar elastic medium.

Let δ be a real number such that $0 < \varepsilon < \delta < \ell/4$. Let $\mathcal{C} \subset \mathbb{R}^3$ be the cylinder $\mathcal{C} := D(0,\delta)\times(0,\ell)$, and $\mathcal{B}^-$, $\mathcal{B}^+$ be the half-balls (cf. Figure 2) defined by

$$\mathcal{B}^- := \{x \in \mathcal{C};\ \|x\| < \delta\} \quad \text{and} \quad \mathcal{B}^+ := \{x \in \mathcal{C};\ \|x-(0,\ell)\| < \delta\}.$$

We denote $\mathfrak{H}$ the set of piecewise-C^1 functions on $\mathcal{C}$ which are constant on $\mathcal{B}^-$ and $\mathcal{B}^+$ (then u^- and u^+ denote respectively the values taken by a function u on $\mathcal{B}^-$ and $\mathcal{B}^+$).

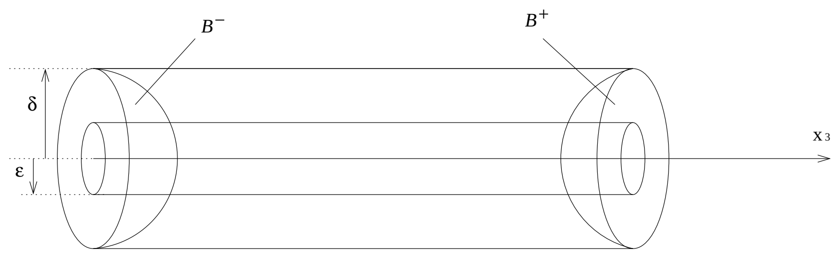

Figure 2: Geometry of the cylinder $\mathcal{C}$.

Lemma 2 *There exists an operator $\Lambda_{\mathcal{C},\varepsilon}$ from $\mathfrak{H}$ to $\mathfrak{H}$, which satisfies, for any $u \in \mathfrak{H}$,*

$$\Lambda_{\mathcal{C},\varepsilon}(u) = u^- \text{ on } \mathcal{B}^- \quad , \quad \Lambda_{\mathcal{C},\varepsilon}(u) = u^+ \text{ on } \mathcal{B}^+, \tag{3.12}$$

$$\Lambda_{\mathcal{C},\varepsilon}(u) = u \quad \text{on } \partial\mathcal{C} \quad , \quad \|\Lambda_{\mathcal{C},\varepsilon}(u)\| \leq c \text{ on } \mathcal{C} \quad , \tag{3.13}$$

$$\int_{D(0,\varepsilon)\times(0,\ell)} j_\nu(\Lambda_{\mathcal{C},\varepsilon}(u))\,dx \leq \frac{\pi\varepsilon^2}{\ell}(1+\nu)(u_3^+ - u_3^-)^2 + c\varepsilon^2\delta, \tag{3.14}$$

$$\int_{(D(0,\delta)\setminus D(0,\varepsilon))\times(0,\ell)} j_\nu(\Lambda_{\mathcal{C},\varepsilon}(u))\,dx \leq c\left(\delta^2 - \varepsilon^2 + \frac{1}{\log(\delta) - \log(\varepsilon)}\right), \tag{3.15}$$

where c is a constant depending only on ν, ℓ and $M := \max(\|u\|_\infty, \|e(u)\|_\infty)$.

Proof: Let us construct a function $\Lambda_{\mathcal{C},\varepsilon}(u)$ (let us simply denote it by v) which satisfies (3.12)-(3.15). Let (e_1, e_2, e_3) be a new orthonormal basis in which e_3 is still directed along the old x_3-axis and $(u^+ - u^-)$ takes the form $(u^+ - u^-) = \xi_1 e_1 + \xi_3 e_3$. We introduce γ in $C^0((0,\ell),(0,1))$ as follows

$$\gamma(s) := 0 \text{ if } s \leq \delta \text{ or } s \geq \ell - \delta, \quad \gamma(s) := 1 \text{ if } s \in (2\delta, \ell - 2\delta),$$

$$\gamma(s) := \frac{s}{\delta} - 1 \text{ if } s \in (\delta, 2\delta) \quad \text{and} \quad \gamma(s) := \frac{\ell - s}{\delta} - 1 \text{ if } s \in (\ell - 2\delta, \ell - \delta).$$

We define v on $D(0,\varepsilon) \times (0,\ell)$ by setting $v(x) := u^-$ if $x_3 \leq \delta$, $v(x) := u^+$ if $x_3 \geq \ell - \delta$ and, for all $x = (x_1, x_2, x_3) \in D(0,\varepsilon) \times [\delta, \ell - \delta]$,

$$\begin{cases} v_1(x) := u_1^- + \xi_1 \left[3\left(\dfrac{x_3 - \delta}{\ell - 2\delta}\right)^2 - 2\left(\dfrac{x_3 - \delta}{\ell - 2\delta}\right)^3\right] - \dfrac{\xi_3 \nu}{\ell}\gamma(x_3)x_1, \\[2ex] v_2(x) := u_2^- - \dfrac{\xi_3 \nu}{\ell}\gamma(x_3)x_2, \\[2ex] v_3(x) := u_3^- - \xi_1 \left[\dfrac{6(x_3 - \delta)}{(\ell - 2\delta)^2} - \dfrac{6(x_3 - \delta)^2}{(\ell - 2\delta)^3}\right] x_1 + \xi_3 \left(\dfrac{x_3 - \delta}{\ell - 2\delta}\right). \end{cases}$$

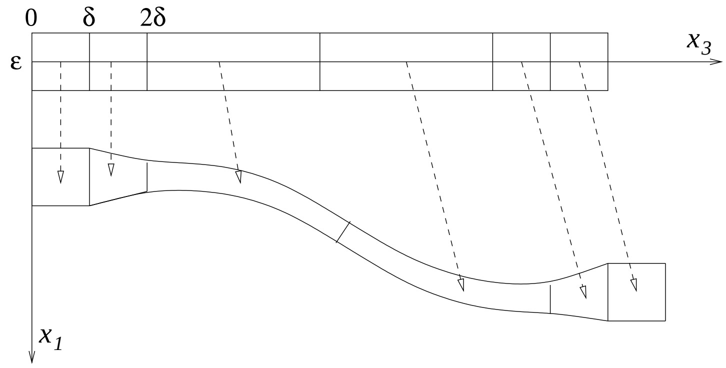

Figure 3: The displacement v in a beam.

From a mechanical point of view, v is an approximation for the motion of an elastic beam which has prescribed constant displacements on each extremity. The terms involving ξ_1 describe the flexion of the beam, the last term in v_3 describes its extension while the terms involving γ describe the constriction (or enlargement if $\nu < 0$) due to a non vanishing Poisson coefficient (see Figure 3). The accuracy of this approximation is stated by the following estimations. Clearly v is piecewise-C^1 on $D(0,\varepsilon)\times(0,\ell)$. Rough estimations of every component of v and ∇v show that there exist constants M_1, M_2, M_3 depending only on M, ν and ℓ such that

$$\|v(x)\| \leq M_1, \quad \|\nabla v(x))\| \leq M_2, \tag{3.16}$$

for all $x \in D(0,\varepsilon) \times (0,\ell)$, and

$$|e_{33}(v)(x) - \ell^{-1}\xi_3| \leq M_3\delta, \tag{3.17}$$

for all $x \in D(0,\varepsilon) \times (\delta, \ell-\delta)$. Using these estimations and Remark 4 we can write

$$\int_{D(0,\varepsilon)\times(0,\ell)} j_\nu(v)\,dx \leq 2K_\nu \pi M_2^2 \varepsilon^2 \delta + \int_{D(0,\varepsilon)\times(2\delta,\ell-2\delta)} j_\nu(v)\,dx.$$

On $D(0,\varepsilon) \times (2\delta, \ell-2\delta)$, a straightforward computation leads to

$$\begin{aligned} j_\nu(v) &= (1+\nu)(e_{33}(v)(x))^2 + \frac{2\nu^2}{1-2\nu}\left(e_{33}(v)(x) - \frac{\xi_3}{\ell}\right)^2 \\ &\leq (1+\nu)\left(\frac{\xi_3}{\ell}\right)^2 + M_4\delta, \end{aligned} \tag{3.18}$$

where M_4 depends only on M, ν and ℓ. Inequality (3.14) follows. It reflects a well known property of thin elastic beams: the energy related to their flexion is negligible when compared with extension.

We extend v on the set $(D(0,\delta) \setminus D(0,\varepsilon)) \times (0,\ell)$ by setting, for any $x' \in D(0,\delta) \setminus D(0,\varepsilon)$ and any $x_3 \in [0,\ell]$,

$$v(x',x_3) = \frac{\log(\|x'\|) - \log(\varepsilon)}{\log(\delta) - \log(\varepsilon)} u(x',x_3) + \frac{\log(\delta) - \log(\|x'\|)}{\log(\delta) - \log(\varepsilon)} v(\frac{\varepsilon x'}{\|x'\|}, x_3).$$

Clearly, v is still piecewise-C^1 on $D(0,\delta)\times(0,\ell)$. Moreover it coincides with u on the "lateral" boundary $\{(x',x_3);\ \|x'\|=\delta\}$ of the cylinder. This is also true on $\mathcal{B}^-$ and $\mathcal{B}^+$ since both $u(x',x_3)$ and $v(\frac{\varepsilon x'}{\|x'\|},x_3)$ coincide with the constants u^- and u^+ when (x',x_3) belongs to these sets. From the definition of v and the estimations (3.16) and (3.17) we get the existence of constants C_1 and C_2, depending only on M, ν and ℓ such that, for any $x'\in D(0,\delta)\setminus D(0,\varepsilon)$ and any $x_3\in(0,\ell)$,

$$\|e(v)(x',x_3)\|^2\le C_1+C_2\left(\frac{1}{\log(\delta)-\log(\varepsilon)}\right)^2\frac{1}{\|x'\|^2}.$$

Using Remark 4, we get by integration

$$\int_{(D(0,\delta)\setminus D(0,\varepsilon))\times(0,\ell)} j_\nu(v)\,dx\le K_\nu\left[\pi C_1(\delta^2-\varepsilon^2)\ell+2\pi C_2\ell\frac{1}{\log(\delta)-\log(\varepsilon)}\right].$$

Inequality (3.15) follows. □

The next lemma estimates the capacity of a net of balls and that of small parts of the balls.

Lemma 3 *Let $\eta<-1$ and $\gamma<3+\eta$ be two real numbers. Let (v^n) be a sequence converging weakly to v in $\mathrm{L}^2(C,\mathbb{R}^3)$.*
i) If, for some $M>0$, v^n satisfies

$$\int_C\|e(v^n)\|^2dx\le Mn^\gamma, \tag{3.19}$$

then the weak convergence in $\mathrm{L}^2(C,\mathbb{R}^3)$ of $\sum_{I\in\mathcal{I}^n}\left(⨍_{B(z_I^n,n^\eta)}v^n(x)\right)\mathbf{1}_{C_I^n}$ to v is assured.
ii) Let $\mathcal{D}_I^n\subset B(z_I^n,n^\eta)$ be centered subsets (i.e. such that $⨍_{\mathcal{D}_I^n}x\,dx=z_I^n$) satisfying $|\mathcal{D}_I^n|\ge s_n>0$ for some sequence (s_n). If v^n satisfies, in addition to (3.19), the following condition

$$\sum_{I\in\mathcal{I}^n}\int_{B(z_I^n,n^\eta)}\|e(v^n)\|^2dx\le Ms_nn^{2-2\eta}, \tag{3.20}$$

then the weak convergence in $\mathrm{L}^2(C,\mathbb{R}^3)$ of $\sum_{I\in\mathcal{I}^n}\left(⨍_{\mathcal{D}_I^n}v^n(x)\,dx\right)\mathbf{1}_{C_I^n}$ to v is also assured.

Proof: In C_I^n, let us consider the sets $E_I^n:=\{x\in C_I^n;\ (8n)^{-1}<\|x-z_I^n\|<(4n)^{-1}\}$. We consider also the functions $\bar v^n$ defined on each C_I^n by

$$\bar v^n(x)=v^n(x)-\left(⨍_{C_I^n}\nabla^a v^n\right)\cdot(x-z_I^n),$$

where $\nabla^a v^n := \nabla v^n - e(v^n)$ denotes the antisymmetric part of the gradient of v^n. We have

$$e(\bar{v}^n) = e(v^n), \quad \fint_{C_I^n} \nabla^a \bar{v}^n = 0, \tag{3.21}$$

$$\fint_{B(z_I^n, n^\eta)} \bar{v}^n = \fint_{B(z_I^n, n^\eta)} v^n, \quad \text{and} \quad \fint_{E_I^n} \bar{v}^n = \fint_{E_I^n} v^n. \tag{3.22}$$

Korn's inequality states the existence of a constant C_0 (invariant by rescaling and independent of n) such that, for every $w \in \mathrm{H}^1(\Omega, \mathbb{R}^3)$,

$$\int_{C_I^n} \left\| \nabla w - \fint_{C_I^n} \nabla^a w \right\|^2 dx \le C_0 \int_{C_I^n} \|e(w)\|^2 \, dx.$$

Applying this inequality to $\bar{v}^n$ we obtain

$$\int_{C_I^n} \|\nabla \bar{v}^n\|^2 \, dx \le C_0 \int_{C_I^n} \|e(v^n)\|^2 \, dx. \tag{3.23}$$

The Poincaré-Wirtinger inequality (and rescaling) states the existence of a constant $C_1 > 0$ such that

$$\int_{C_I^n} \left\| v^n - \fint_{E_I^n} v^n \right\|^2 dx \le \frac{C_1}{n^2} \int_{C_I^n} \|\nabla v^n\|^2 \, dx. \tag{3.24}$$

Let S^2 be the unit sphere of $\mathbb{R}^3$ and let us use in each C_I^n spherical coordinates with center $z_I^n : \rho := \|x - z_I^n\|$, $y := \rho^{-1}(x - z_I^n) \in S^2$. For all $0 < a < b < (4n)^{-1}$, we have

$$\int_{C_I^n} \|\nabla \bar{v}^n\|^2 \, dx \ge \int_{S^2} \int_a^b \left\| \frac{\partial}{\partial \rho} (\bar{v}^n (z_I^n + \rho y)) \right\|^2 \rho^2 d\rho \, dy.$$

A simple one-dimensional minimization shows that

$$ab^2 \int_a^b \left\| \frac{\partial \bar{v}^n}{\partial \rho} (z_I^n + \rho y) \right\|^2 \rho^2 d\rho \ge a^2 b^2 \left\| \bar{v}^n (z_I^n + by) - \bar{v}^n (z_I^n + ay) \right\|^2.$$

Hence

$$\int_{S^2} a^2 b^2 \left\| \bar{v}^n (z_I^n + by) - \bar{v}^n (z_I^n + ay) \right\|^2 dy \le ab^2 \int_{C_I^n} \|\nabla \bar{v}^n\|^2 \, dx.$$

Integrating this inequality for $a \in (0, n^\eta)$ and $b \in (\frac{1}{8n}, \frac{1}{4n})$ and using the Holder inequality we get

$$\left\| \int_0^{n^\eta} \int_{\frac{1}{8n}}^{\frac{1}{4n}} \int_{S^2} a^2 b^2 \left(\bar{v}^n(z_I^n + by) - \bar{v}^n(z_I^n + ay) \right) dy\, db\, da \right\|^2 \leq \left(\int_0^{n^\eta} \int_{\frac{1}{8n}}^{\frac{1}{4n}} ab^2 db\, da \right) \times \left(\int_0^{n^\eta} \int_{\frac{1}{8n}}^{\frac{1}{4n}} \int_{S^2} a^2 b^2 dy\, db\, da \right) \times \left(\int_{C_I^n} \|\nabla \bar{v}^n\|^2 dx \right). \qquad (3.25)$$

Thus

$$\left\| ⨍_{E_I^n} \bar{v}^n - ⨍_{B(z_I^n, n^\eta)} \bar{v}^n \right\|^2 \leq \frac{3}{8\pi n^\eta} \int_{C_I^n} \|\nabla \bar{v}^n\|^2 dx. \qquad (3.26)$$

Using (3.22) and (3.23) we obtain

$$\left\| ⨍_{E_I^n} v^n - ⨍_{B(z_I^n, n^\eta)} v^n \right\|^2 \leq \frac{3C_0}{8\pi n^\eta} \int_{C_I^n} \|e(v^n)\|^2 dx. \qquad (3.27)$$

This estimation, together with (3.24), leads to

$$\int_{C_I^n} \left\| v^n - ⨍_{B(z_I^n, n^\eta)} v^n \right\|^2 dx \leq \frac{2C_1}{n^2} \int_{C_I^n} \|\nabla v^n\|^2 dx + \frac{3C_0}{4\pi n^{3+\eta}} \int_{C_I^n} \|e(v^n\|^2 dx.$$

Summing over $I \in \mathcal{I}^n$, we get

$$\int_C \left\| v^n - \sum_{I \in \mathcal{I}^n} \left(⨍_{B(z_I^n, n^\eta)} v^n \right) \mathbf{1}_{C_I^n} \right\|^2 dx \leq \frac{2C_1}{n^2} \int_C \|\nabla v^n\|^2 dx + \frac{3C_0}{4\pi n^{3+\eta}} \int_C \|e(v^n)\|^2 dx. \qquad (3.28)$$

Assertion i) of the lemma is proved since its assumptions and Korn's inequality assure that every term of the right hand side of the last inequality tends to zero.

We prove the second assertion in a similar way: we consider $(\tilde{v}^n)$ defined on each $B(z_I^n, n^\eta)$ by

$$\tilde{v}^n(x) = v^n(x) - \left(⨍_{B(z_I^n, n^\eta)} \nabla^a v^n \right) \cdot (x - z_I^n).$$

As z_i^n is the center of both $B(z_I^n, n^\eta)$ and $\mathcal{D}_I^n$, we have

$$e(\tilde{v}^n) = e(v^n), \quad \fint_{B(z_I^n,n^\eta)} \nabla^a \tilde{v}^n = 0,$$
$$\fint_{B(z_I^n,n^\eta)} \tilde{v}^n = \fint_{B(z_I^n,n^\eta)} v^n, \quad \text{and} \quad \fint_{\mathcal{D}_I^n} \tilde{v}^n = \fint_{\mathcal{D}_I^n} v^n.$$

The Korn inequality states the existence of a constant C_2 (invariant by rescaling and then independent of n) such that, for any $w \in \mathrm{H}^1(\Omega, \mathbb{R}^3)$,

$$\int_{B(z_I^n,n^\eta)} \left\| \nabla w - \fint_{B(z_I^n,n^\eta)} \nabla^a w \right\|^2 dx \le C_2 \int_{B(z_I^n,n^\eta)} \|e(w)\|^2 \, dx.$$

Moreover, denoting by C_3 the Poincaré's constant of the unit ball of $\mathbb{R}^3$, we get by rescaling

$$\int_{B(z_I^n,n^\eta)} \left\| w - \fint_{B(z_I^n,n^\eta)} w \right\|^2 dx \le C_3 n^{2\eta} \int_{B(z_I^n,n^\eta)} \|\nabla w\|^2 \, dx.$$

Applying these last two inequalities to $(\tilde{v}^n)$, we obtain

$$\begin{aligned}\int_{\mathcal{D}_I^n} \left\| \tilde{v}^n - \fint_{B(z_I^n,n^\eta)} \tilde{v}^n \right\|^2 dx &\le \int_{B(z_I^n,n^\eta)} \left\| \tilde{v}^n - \fint_{B(z_I^n,n^\eta)} \tilde{v}^n \right\|^2 dx \\ &\le C_2 C_3 n^{2\eta} \int_{B(z_I^n,n^\eta)} \|e(\tilde{v}^n)\|^2 \, dx.\end{aligned}$$

and so

$$\begin{aligned}\left\| \fint_{\mathcal{D}_I^n} v^n - \fint_{B(z_I^n,n^\eta)} v^n \right\|^2 &\le \frac{C_2 C_3 n^{2\eta}}{|\mathcal{D}_I^n|} \int_{B(z_I^n,n^\eta)} \|e(v^n)\|^2 \, dx \\ &\le \frac{C_2 C_3 n^{2\eta}}{s_n} \int_{B(z_I^n,n^\eta)} \|e(v^n)\|^2 \, dx. \qquad (3.29)\end{aligned}$$

Summing over all $I \in \mathcal{I}^n$ we get

$$\left\| \sum_{I \in \mathcal{I}^n} \left(\fint_{\mathcal{D}_I^n} v^n(x) \right) \mathbf{1}_{C_I^n} - \sum_{I \in \mathcal{I}^n} \left(\fint_{B(z_I^n,n^\eta)} v^n(x) \right) \mathbf{1}_{C_I^n} \right\|^2_{\mathrm{L}^2(C,\mathbb{R}^3)} \le \frac{M C_2 C_3}{n}.$$

The proof of assertion (ii) is then concluded by using assertion (i). □

The next lemma enables us to approximate a regular function on C by a function taking constant values on a net of balls.

Lemma 4 *For any integer n, any real number $\eta \leq -2$, any function $u \in C^1(C, \mathbb{R}^3)$ and any family $(w_I^n)_{I \in \mathcal{I}^n} \in (\mathbb{R}^3)^{n^3}$, let us set*

$$T_{n,\eta}(u,(w_I^n))(x) := \begin{cases} w_I^n, & \text{if } x \in B(z_I^n, n^\eta), \\ u(x), & \text{if } \forall I \in \mathcal{I}^n,\ \|x - z_I^n\| \geq 2n^{-2}, \\ \dfrac{2\|x-z_I^n\| - 2n^\eta}{(2-n^{\eta+2})\,\|x-z_I^n\|}\, u(x) + \dfrac{2n^\eta - n^{\eta+2}\|x-z_I^n\|}{(2-n^{\eta+2})\,\|x-z_I^n\|}\, w_I^n, & \\ & \text{if } n^\eta \leq \|x-z_I^n\| \leq 2n^{-2}. \end{cases}$$

The operator $T_{n,\eta}$ satisfies

i) $T_{n,\eta}(u,(w_I^n))$ is piecewise-C^1, coincides with u on the boundary of C,

ii) $\|T_{n,\eta}(u,(w_I^n))\|_\infty \leq \max(\|u\|_\infty, \max_I \|w_I^n\|)$,

iii) $\int_C \|e(T_{n,\eta}(u,(w_I^n)))\|^2 dx < c\, n^{(3+\eta)^+}$, where $(3+\eta)^+ := \max(0, 3+\eta)$ and c depends only on $\max(\|u\|_\infty, \|\nabla u\|_\infty, \max_I \|w_I^n\|)$,

iv) if $\eta < -3$ and if the families (w_I^n) are uniformly bounded, then the sequence $T_{n,\eta}(u,(w_I^n))$ converges strongly to u in $\mathrm{H}^1(C,\mathbb{R}^3)$ as n tends to infinity,

v) $\|\nabla T_{n,\eta}(u,(u(z_I^n)))\|_\infty \leq 10\|\nabla u\|_\infty$,

vi) $T_{n,\eta}(u,(u(z_I^n)))$ converges strongly to u in $\mathrm{H}^1(C,\mathbb{R}^3)$ as n tends to infinity.

Proof: Assertion (i) is a simple consequence of assumption (3.6) which, for $\eta \leq -2$, prevents the balls $B(z_I^n, n^\eta)$ to intersect each other and with the boundary of C.

Assertion (ii) is obvious since, at any point $x \in C$, $T_{n,\eta}(u,(w_I^n))(x)$ is either $u(x)$, some w_I^n, or an interpolation between $u(x)$ and some w_I^n.

In each C_I^n, let us set $\rho := \|x - z_I^n\|$ and denote by $\mathcal{G}_I^n$ the transition zone $\mathcal{G}_I^n := \{x \in C_I^n;\ \rho \in (n^\eta, 2n^{-2})\}$. For any $x \in \mathcal{G}_I^n$, we have

$$\|\nabla(T_{n,\eta}(u,(w_I^n))(x)\|^2 \leq 2\|\nabla u(x)\|^2 + 8\|u(x) - w_I^n\|^2 n^{2\eta}\rho^{-4}. \qquad (3.30)$$

Hence

$$\sum_{I \in \mathcal{I}^n} \int_{\mathcal{G}_I^n} \|e(T_{n,\eta}(u,(w_I^n))(x)\|^2\, dx \leq 64\pi \left(\frac{1}{3}\|\nabla u\|_\infty^2 n^{-3} + (\max(\|u\|_\infty, \max_I \|w_I^n\|))^2 n^{3+\eta}\right) \qquad (3.31)$$

Assertion (iii) follows, since, outside of the transition zones $\mathcal{G}_I^n$, $T_{n,\eta}(u,(w_I^n))$ is either constant or coincides with u.

The domain where $T_{n,\eta}(u,(w_I^n))$ does not coincide with u has a volume tending to zero. Owing to (ii), we know that $T_{n,\eta}(u,(w_I^n))$ converges to u strongly in $\mathrm{L}^2(C,\mathbb{R}^3)$. Inequality (3.31), when $\eta < -3$, implies

$$\int_C \|e(u - T_{n,\eta}(u,(w_I^n))(x)\|^2\,dx \to 0.$$

Assertion (iv) is proved invoking Korn's inequality.

In the particular case $w_I^n = u(z_I^n)$, inequality (3.30) leads to the finer estimation:

$$\begin{aligned}\|\nabla(T_{n,\eta}(u,(u(z_I^n)))(x)\|^2 &\le 2\|\nabla u(x)\|^2 + 8\|\nabla u\|_\infty^2 n^{2\eta}\rho^{-2}\\ &\le 10\|\nabla u\|_\infty^2. \end{aligned} \tag{3.32}$$

Assertion (v) is proved.

The difference $u - T_{n,\eta}(u,(u(z_I^n))$ vanishes outside a domain the volume of which tends to zero. It is uniformly bounded and so is its gradient. Assertion (vi) follows. □

3.2.2. Lower-bound inequality Let (u^n) be a sequence with bounded energy ($E_{\alpha_n,\nu}(u^n) < M$) and converging weakly to some u in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$. We have

$$E_{\alpha_n,\nu}(u^n) \ge \int_{\Omega^n} \alpha_n j_\nu(u^n)\,dx \ge n^{-3}r_n^{-2}\int_{\Omega^n} j_\nu(u^n)\,dx. \tag{3.33}$$

Denoting temporarily $a := n^{-2}$, $b := \|x_2 - x_1\|$, $\ell := \|x_2 - x_1\| + n^{-2}$ and $w := \frac{x_2 - x_1}{\|x_2 - x_1\|}$, we apply Lemma 1 to each cylinder

$$z_I^n + x_1 + \left\{x' + tw\ ;\ x' \perp w,\ \|x'\| < r_I^n,\ t \in \left(-\frac{a}{2}, \ell - \frac{a}{2}\right)\right\}, \tag{3.34}$$

and noticing that these cylinders are disjoint and contained in Ω^n, we get

$$\begin{aligned}E_{\alpha_n,\nu}(u^n) \ge \frac{1+\nu}{n^3 r_n^2} \sum_{I\in\mathcal{I}^n} &\frac{\pi {r_I^n}^2}{\|x_2 - x_1\| + n^{-2}}\\ &\times\left[\left(⨍_{x_2+\mathcal{D}_I^n} u^n - ⨍_{x_1+\mathcal{D}_I^n} u^n\right)\cdot\frac{x_2 - x_1}{\|x_2 - x_1\|}\right]^2, \end{aligned} \tag{3.35}$$

where $\mathcal{D}_I^n$ denotes the cylinder (included in $\mathcal{B}_I^n$) defined by

$$\mathcal{D}_I^n := z_I^n + \left\{x' + tw\ ;\ x' \perp w,\ \|x'\| < r_I^n,\ t \in \left(-\frac{a}{2}, \frac{a}{2}\right)\right\}, \tag{3.36}$$

so that $x_1 + \mathcal{D}_I^n$ and $x_2 + \mathcal{D}_I^n$ are the extremity parts of the cylinders defined by (3.34). Recalling Definition (3.5) of r_I^n, we obtain

$$\begin{aligned}E_{\alpha_n,\nu}(u^n) \ge &\frac{\|x_2 - x_1\|}{\|x_2 - x_1\| + n^{-2}} \sum_{I\in\mathcal{I}^n}\left(\int_{C_I^n} g(x)\,dx\right)\\ &\times\left[\left(⨍_{x_2+\mathcal{D}_I^n} u^n - ⨍_{x_1+\mathcal{D}_I^n} u^n\right)\cdot\frac{x_2 - x_1}{\|x_2 - x_1\|}\right]^2 .\end{aligned} \tag{3.37}$$

Hence

$$E_{\alpha_n,\nu}(u^n) \geq \frac{\|x_2 - x_1\|}{\|x_2 - x_1\| + n^{-2}} \int_C \Bigg[\Bigg(\Bigg(\sum_{I \in \mathcal{I}^n} \Big(⨍_{x_2 + \mathcal{D}_I^n} u^n \Big) \mathbf{1}_{C_I^n}(x) \Bigg) - \Bigg(\sum_{I \in \mathcal{I}^n} \Big(⨍_{x_1 + \mathcal{D}_I^n} u^n \Big) \mathbf{1}_{C_I^n}(x) \Bigg) \Bigg) \cdot \frac{x_2 - x_1}{\|x_2 - x_1\|} \Bigg]^2 g(x)\, dx. \quad (3.38)$$

Definition (3.5) of r_I^n ensures that the volume of every cylinder $\{\mathcal{D}_I^n\}$ is larger than $s_n := \pi r_n^2 n^{-1/2} n^{-2}$. For $i = 1$ or 2, $\int_C \alpha_n(x_i + x)\|e(u^n)(x_i + x)\|^2\, dx \leq M$ implies

$$\int_C \|e(u^n)(x_i + x)\|^2\, dx \leq M n^{1/2}$$

and

$$\sum_{I \in \mathcal{I}^n} \int_{\mathcal{B}_I^n} \|e(u^n)(x_i + x)\|^2\, dx \leq M n^3 r_n^2 = \frac{M}{\pi} s_n n^{11/2}.$$

Then assumptions of Lemma 3 are satisfied if we set $\eta = -2$ and $\gamma = 1/2$. Indeed the translated functions $x \to u^n(x - x_2)$ and $x \to u^n(x - x_1)$ converge weakly in $\mathrm{L}^2(C, \mathbb{R}^3)$ to the corresponding translated functions of u. Lemma 3 allows us to pass to the limit in inequality (3.38) and we get the lower-bound

$$\begin{aligned} \liminf_{n \to \infty} E_{\alpha_n,\nu}(u^n) &\geq \int_C \left[(u(x_2 + x) - u(x_1 + x)) \cdot \frac{x_2 - x_1}{\|x_2 - x_1\|} \right]^2 g(x)\, dx \\ &= S_{(x_1,x_2),g}(u). \end{aligned} \quad (3.39)$$

□

3.2.3. Upperbound inequality For any $\mathrm{w} \in \mathbb{R}^3$ and any function u, we denote by $\tau_{\mathrm{w}} u$ the translated function: $\tau_{\mathrm{w}} u(x) := u(x - \mathrm{w})$. Let $u \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$ such that $S_{(x_1,x_2),g}(u) < +\infty$. By a density argument we can assume that u belongs to $C^\infty(\Omega, \mathbb{R}^3)$. Let us first apply Lemma 4 and define, for all $x \in \Omega$,

$$\begin{cases} \widetilde{u}^n(x) = T_{n,-2}(\tau_{-x_1} u, (u(x_1 + x_I^n)))(x - x_1) & \text{if } x \in x_1 + C, \\ \widetilde{u}^n(x) = T_{n,-2}(\tau_{-x_2} u, (u(x_2 + x_I^n)))(x - x_2) & \text{if } x \in x_2 + C, \\ \widetilde{u}^n(x) = u(x) & \text{otherwise.} \end{cases} \quad (3.40)$$

From Lemma 4, we know that $\widetilde{u}^n$ is piecewise-C^1 and constant on the balls $x_1 + \mathcal{B}_I^n$ and $x_2 + \mathcal{B}_I^n$. There exists a constant M, depending only on u, such that, for all $x \in \Omega$,

$$\|\widetilde{u}^n(x)\| < M \quad \text{and} \quad \|e(\widetilde{u}^n)(x)\| < M. \quad (3.41)$$

Moreover $\widetilde{u}^n$ converges to u strongly in $\mathrm{H}^1(\Omega, \mathbb{R}^3)$.

For each I in $\mathcal{I}^n$, using adapted cylindrical coordinates (x', x_3) of center $z_I^n + x_1$ and axis $x_2 - x_1$, the fiber $\mathcal{F}_I^n$ reads

$$\mathcal{F}_I^n = \{(x', x_3);\ \|x'\| \le r_I^n,\ x_3 \in (0, \|x_2 - x_1\|)\}\,.$$

It is contained in the larger cylinder

$$\mathcal{A}_I^n = \left\{(x', x_3);\ \|x'\| \le n^{-2},\ x_3 \in (0, \|x_2 - x_1\|)\right\}.$$

Lemma 2 provides on $\mathcal{A}_I^n$ a function $\Lambda_{\mathcal{A}_I^n, r_I^n}(\widetilde{u}^n)$ satisfying

$$\int_{\mathcal{F}_I^n} j_\nu(\Lambda_{\mathcal{A}_I^n, r_I^n}(\widetilde{u}^n))(x)\,dx \le \frac{\pi (r_I^n)^2}{\|x_2 - x_1\|}(1+\nu)(u_3^+ - u_3^-)^2 + c'(r_I^n)^2 n^{-2}, \tag{3.42}$$

and

$$\int_{\mathcal{A}_I^n \setminus \mathcal{F}_I^n} j_\nu(\Lambda_{\mathcal{A}_I^n, r_I^n}(\widetilde{u}^n))(x)\,dx \le c'\left(n^{-4} - (r_I^n)^2 - \frac{1}{2\log(n) + \log(r_I^n)}\right). \tag{3.43}$$

where c' depends only on x_1, x_2, ν and M (i.e. on u). Owing to assumption (3.7), the cylinders $\mathcal{A}_I^n$ are disjoint and we can define the approximating sequence u^n by:

$$\begin{cases} u^n(x) = \Lambda_{\mathcal{A}_I^n, r_I^n}(\widetilde{u}^n)(x) & \text{if } x \in \mathcal{A}_I^n, \\ u^n(x) = \widetilde{u}^n(x) & \text{otherwise.} \end{cases} \tag{3.44}$$

Let us now estimate $E_{\alpha_n,\nu}(u^n)$. Note first that u^n is constant on the balls $x_1 + \mathcal{B}_I^n$ and $x_2 + \mathcal{B}_I^n$. Indeed, owing to (3.12), it coincides with $\widetilde{u}^n$ on these balls. Hence

$$E_{\alpha_n,\nu}(u^n) = n^{-1/2} \int_{\Omega \setminus \cup_{I \in \mathcal{I}^n} \mathcal{F}_I^n} j_\nu(u^n)\,dx + n^{-3} r_n^{-2} \int_{\cup_{I \in \mathcal{I}^n} \mathcal{F}_I^n} j_\nu(u^n)\,dx. \tag{3.45}$$

Inequality (3.41) assures that

$$\limsup_{n \to \infty} \left(n^{-1/2} \int_{\Omega \setminus \cup_{I \in \mathcal{I}^n} \mathcal{A}_I^n} j_\nu(u^n)\,dx \right) = 0. \tag{3.46}$$

Summing inequalities (3.43) for all $I \in \mathcal{I}^n$ and taking into account the order of magnitude of r_n given by (3.3), we get

$$\limsup_{n \to \infty} \left(\int_{\cup_{I \in \mathcal{I}^n} (\mathcal{A}_I^n \setminus \mathcal{F}_I^n)} j_\nu(u^n)(x)\,dx \right) = 0, \tag{3.47}$$

and obviously

$$\limsup_{n \to \infty} \left(n^{-1/2} \int_{\cup_{I \in \mathcal{I}^n} (\mathcal{A}_I^n \setminus \mathcal{F}_I^n)} j_\nu(u^n)(x)\,dx \right) = 0. \tag{3.48}$$

Finally, summing inequalities (3.42) for all $I \in \mathcal{I}^n$ and recalling Definition (3.5) of r_I^n, we get

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\cup_{I\in\mathcal{I}^n}\mathcal{F}_I^n} n^{-3} r_n^{-2} j_\nu(u^n)(x)\,dx \\
&\le \limsup_{n\to\infty} \sum_{I\in\mathcal{I}^n} \frac{r_n^{-2} n^{-3}(1+\nu)\pi(r_I^n)^2}{\|x_2-x_1\|}\Big[\big(u(x_1+z_I^n)-u(x_2+z_I^n)\big)\cdot\frac{x_2-x_1}{\|x_2-x_1\|}\Big]^2 \\
&\le \limsup_{n\to\infty} \sum_{I\in\mathcal{I}^n} \int_{C_I^n} \Big[\big(u(x_1+z_I^n)-u(x_2+z_I^n)\big)\cdot\frac{x_2-x_1}{\|x_2-x_1\|}\Big]^2 g(x)\,dx \\
&\le \int_\Omega \Big[\big(u(x_1+x)-u(x_2+x)\big)\cdot\frac{x_2-x_1}{\|x_2-x_1\|}\Big]^2 g(x)\,dx. \qquad (3.49)
\end{aligned}$$

This inequality, together with (3.45), (3.46) and (3.48) proves that u^n satisfies the upper-bound inequality

$$\limsup_{n\to\infty} E_{\alpha_n,\nu}(u^n) \le S_{(x_1,x_2),g}(u). \qquad (3.50)$$

From (3.47), (3.49) and Remark 4 we deduce also that

$$\limsup_{n\to\infty}\int_{\cup_{I\in\mathcal{I}^n}\mathcal{A}_I^n} \|e(u^n)\|^2\,dx \le k_\nu^{-1}\limsup_{n\to\infty}\int_{\cup_{I\in\mathcal{I}^n}\mathcal{A}_I^n} j_\nu(u^n)\,dx = 0. \qquad (3.51)$$

and then that $u^n - \widetilde{u}^n$ converges strongly to zero in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. Thus u^n like $\widetilde{u}^n$ tends to u strongly in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$. This, together with the lower-bound (3.39) and the upper-bound (3.50) concludes the proof of the τ-convergence theorem (Theorem 3) and then the proof of Theorem 2. □

4. Multipoint non-local interactions

In this section we extend the result of the previous section to interactions involving more than two points, namely proving that, for all positive integers p, $\mathfrak{L}_p$ and more generally $\mathfrak{Q}_p$ are included in the closure of elasticity functionals $\mathfrak{E}_\nu$. Next subsection is devoted to the proof of a theorem which is an essential tool for this aim.

4.1. A second homogenization result

Theorem 4 *Let $0 < r < p$ be two integers. Let $(x_i) \in (\mathbb{R}^3)^p$ be a family of p distinct points in Ω and $(y_i) \in (\mathbb{R}^3)^r$ the sub-family $y_i = x_i$, $i = 1\ldots r$. Let q be a non-negative quadratic form on $(\mathbb{R}^3)^p$ and define the non-negative quadratic form $\widetilde{q}$ on $(\mathbb{R}^3)^r$, by setting, for any $(u_i) \in (\mathbb{R}^3)^r$,*

$$\widetilde{q}(u_1,u_2,\ldots,u_r) := \inf\left\{q(u_1,u_2,\ldots,u_p);\ (u_{r+1},u_{r+2},\ldots,u_p)\in(\mathbb{R}^3)^{p-r}\right\}$$

If, for any g in $L^\infty(\mathbb{R}^3,\mathbb{R}^+)$, the functionals $D_{q,(x_i),g}$ belong to the closure $\overline{\mathfrak{E}_\nu}$ of elasticity functionals, then so do the functionals $D_{\widetilde{q},(y_i),g}$.

Proof: For the same reasons as in the proof of Theorem 2, we need only to consider the case when the support of the function g is included in a cube C such that the sets $x_i + C$, $(i \in \{1, \ldots, p\})$ are disjoint and included in Ω.

Let us use the same geometric notation as in section 3.1 and define α_n and g_n in $\mathrm{L}^\infty(\Omega, \mathbb{R})$ by

$$\begin{cases} \alpha_n(x) = n^6, & \text{if } x \in x_i + B(z_I^n, n^{-2}),\ i \in \{1, \ldots, r\},\ I \in \mathcal{I}^n, \\ \alpha_n(x) = n^{-1/2}, & \text{otherwise.} \end{cases} \tag{4.1}$$

and

$$g_n = \sum_{I \in \mathcal{I}^n} \frac{1}{|B(z_I^n, n^{-4})|} \left(\int_{C_I^n} g(x)\, dx \right) \mathbf{1}_{B(z_I^n, n^{-4})}. \tag{4.2}$$

Let us set

$$E_n := D_{q,(x_i),g_n} + E_{\alpha_n,\nu}. \tag{4.3}$$

Assumptions of Theorem 4 imply that, for all n, $D_{q,(x_i),g_n}$ belongs to $\overline{\mathfrak{E}_\nu}$. Owing to Property 3, E_n belongs also to $\overline{\mathfrak{E}_\nu}$. Then, owing to Property 2, the theorem will be proved once it is proved that E_n τ-converges to $D_{\widetilde{q},(y_i),g}$.

4.1.1. Lower-bound Let (u^n) be a sequence converging to u weakly in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ with bounded energy $(E_n(u^n) < M)$. We have

$$\begin{aligned} E_n(u^n) &\geq \int_C q(u^n(x_1 + x), \ldots, u^n(x_p + x))\, g_n(x)\, dx \\ &\geq \int_C \inf_{u_{r+1}, \ldots, u_p} q(u^n(x_1 + x), \ldots, u^n(x_r + x), u_{r+1}, \ldots, u_p)\, g_n(x)\, dx \\ &\geq \sum_{I \in \mathcal{I}^n} \int_{C_I^n} \widetilde{q}(u^n(x_1 + x), \ldots, u^n(x_r + x))\, g_n(x)\, dx \\ &\geq \sum_{I \in \mathcal{I}^n} \left(\int_{C_I^n} g(x)\, dx \right) \left(⨍_{B(z_I^n, n^{-4})} \widetilde{q}(u^n(x_1 + x), \ldots, u^n(x_r + x))\, dx \right), \end{aligned}$$

and by convexity

$$\begin{aligned} E_n(u^n) \geq \sum_{I \in \mathcal{I}^n} \Bigg[\widetilde{q} \left(⨍_{B(z_I^n, n^{-4})} u^n(x_1 + x)\, dx, \ldots, ⨍_{B(z_I^n, n^{-4})} u^n(x_r + x))\, dx \right) \\ \times \left(\int_{C_I^n} g(x)\, dx \right) \Bigg]. \end{aligned} \tag{4.4}$$

If we define $\bar{u}^n$ on each $x_i + C$ $(i = 1, \ldots, r)$ by setting, for all $i \in \{1, \ldots, r\}$ and $x \in C$,

$$\bar{u}^n(x_i + x) := \sum_{I \in \mathcal{I}^n} \left(⨍_{B(z_I^n, n^{-4})} u^n(x_i + y)\, dy \right) \mathbf{1}_{C_I^n}(x) \tag{4.5}$$

and $\bar{u}^n(x) := u^n(x)$ elsewhere, inequality (4.4) reads

$$E_n(u^n) \geq \sum_{I\in\mathcal{I}^n} \int_{C_I^n} \widetilde{q}\left(\bar{u}^n(x_1+x),\ldots,\bar{u}^n(x_r+x)\right) g(x)\,dx$$
$$\geq D_{\widetilde{q},(y_i),g}(\bar{u}^n). \tag{4.6}$$

On the other hand, the bound $E_n(u^n) < M$, implies that, for any $i \in \{1,\ldots,r\}$,

$$n^{-1/2}\int_{x_i+C} \|e(u^n)\|^2\,dx \leq k_\nu^{-1}M$$

and

$$\sum_{I\in\mathcal{I}^n} \int_{B(z_I^n,n^{-4})} n^6\|e(u^n)(x_i+x)\|^2\,dx \leq k_\nu^{-1}M.$$

Assumptions of Lemma 3 are satisfied if we set $\eta = -2$, $\gamma = 1/2$, $\mathcal{D}_I^n = B(z_I^n, n^{-4})$ and $s_n = 4/3\pi n^{-12}$. Thus, for $i \in \{1,\ldots,r\}$, $\bar{u}^n$ converges to u weakly in $\mathrm{L}^2(x_i+C,\mathbb{R}^3)$, $\bar{u}^n$ converges to u weakly in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ and we can pass to the limit in inequality (4.6):

$$\liminf_{n\to\infty} E_n(u^n) \geq D_{\widetilde{q},(y_i),g}(u). \tag{4.7}$$

□

4.1.2. Upper-bound Let $u \in \mathrm{L}^2(\Omega,\mathbb{R}^3)$. By a density argument we can assume that u belongs to $C^\infty(\Omega,\mathbb{R}^3)$. For any $I \in \mathcal{I}^n$ and $i \in \{1,\ldots,r\}$, we set $u_{I,i}^n := u(x_i + z_I^n)$. Then we define $(u_{I,r+1}^n,\ldots,u_{I,p}^n)$ as a solution of the minimization problem

$$\inf\left\{q\big(u_{I,1}^n,\ldots,u_{I,r}^n,u_{I,r+1}^n,\ldots,u_{I,p}^n\big);\ (u_{I,r+1}^n,\ldots,u_{I,p}^n)\in(\mathbb{R}^3)^{p-r}\right\}.$$

We can moreover impose on the chosen solution to satisfy, for some constant M depending only on u and q,

$$\forall I\in\mathcal{I}^n,\ \forall i\in\{1,\ldots,p\}\quad \|u_{I,i}^n\|\leq M.$$

Let us now define an approximating sequence u^n, using the operator $T_{n,\eta}$ provided by Lemma 4, by setting

$$u^n(x) := \begin{cases} \tau_{x_i}T_{n,-2}(\tau_{-x_i}(u),(u_{I,i}^n)) & \text{if } x\in x_i+C,\ i\in\{1,\ldots,r\},\\ \tau_{x_i}T_{n,-4}(\tau_{-x_i}(u),(u_{I,i}^n)) & \text{if } x\in x_i+C,\ i\in\{r+1,\ldots,p\},\\ u(x) & \text{otherwise.}\end{cases} \tag{4.8}$$

Property (vi) of $T_{n,\eta}$ stated in Lemma 4 ensures that u^n converges to u strongly in $\mathrm{H}^1(x_i+C,\mathbb{R}^3)$ for any $i\in\{1,\ldots,p\}$. Hence u^n converges to u strongly in $\mathrm{H}^1(\Omega,\mathbb{R}^3)$.

Property (iii) of $T_{n,\eta}$ stated in Lemma 4 ensures that $\int_{x_i+C} j_\nu(u^n)\,dx$ is bounded for any $i\in\{r+1,\ldots,p\}$ by some M' depending only on u and

M. Property (v) ensures also that $\|\nabla u^n\|$ is uniformly bounded on each set $x_i + C$ when $i \in \{1, \dots, r\}$. Thus

$$\limsup_{n\to\infty} \int_\Omega n^{-1/2} j_\nu(u^n)\, dx \le \limsup_{n\to\infty} K_\nu \int_\Omega n^{-1/2} \|e(u^n)\|, dx = 0. \tag{4.9}$$

As u^n is constant on the balls $x_i + B(z_I^n, n^{-2})$ for $i \in \{1, \dots r\}$ and $I \in \mathcal{I}^n$, the last inequality implies

$$\lim_{n\to\infty} E_{\alpha_n,\nu}(u^n) = 0 \tag{4.10}$$

On the other hand, we have

$$\begin{aligned} D_{q,(x_i),g_n}(u^n) &= \sum_{I\in\mathcal{I}^n} q(u^n_{I,1}, \dots, u^n_{I,p}) \int_{C_I^n} g(x)\, dx \\ &= \sum_{I\in\mathcal{I}^n} \widetilde{q}(u^n_{I,1}, \dots, u^n_{I,r}) \int_{C_I^n} g(x)\, dx. \end{aligned} \tag{4.11}$$

Denoting, as previously, $\bar{u}^n$ the function defined by (4.5) on each $x_i + C$ and $\bar{u}^n(x) := u^n(x)$ elsewhere, the last equality can be rewritten

$$D_{q,(x_i),g_n}(u^n) = D_{\widetilde{q},(y_i),g}(\bar{u}^n). \tag{4.12}$$

Clearly, the regularity of u ensures that $\bar{u}^n$ tends to u strongly in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$. Thus we can pass to the limit and we get

$$\limsup_{n\to\infty} D_{q,(x_i),g_n}(u^n) \le D_{\widetilde{q},(y_i),g}(u). \tag{4.13}$$

Collecting inequalities (4.10) and (4.13) leads to the upper-bound inequality:

$$\limsup_{n\to\infty} E_n(u^n) \le D_{\widetilde{q},(y_i),g}(u). \tag{4.14}$$

This, together with the lower-bound (4.7) concludes the proof of the τ-convergence of E_n to $D_{\widetilde{q},(y_i),g}$ and then the proof of Theorem 4. □

4.2. From elementary to multipoint interactions

In this section we prove that any functional in $\mathfrak{L}_p$ (i.e. of type (2.18)) belongs to the closure of the set of elasticity functionals. This is an extension of Theorem 2 which states this result in the case $p = 2$. For $p > 4$ the result can be obtained by using an induction argument while the cases $p = 3$ and $p = 4$ need a special approach. Then we extend this result to any discrete objective functional, proving that $\mathfrak{O}_p$ is also included in the closure of the set of elasticity functionals. From a mechanical point of view, these results show that any objective interaction between p points can be considered as the interaction due to a "truss", that is a finite union of springs.

Lemma 5 *We have* $\mathfrak{L}_4 \subset \overline{\mathfrak{E}_\nu}$

Proof: Consider a balanced system of four forces (f_i, x_i) $(i = 0, \ldots, 3)$, the distinct points x_i belonging to Ω. Let us first consider the case when the family $\{f_i, i = 1,2,3\} \cup \{x_i - x_0, i = 1,2,3\}$ has rank three. Then there exist $\varepsilon > 0$ such that the points (x_0, x_1', x_2', x_3') (where $x_i' := x_i + \varepsilon f_i$) are not coplanar[2]. Let us define the quadratic form q on $(\mathbb{R}^3)^7$ by setting:

$$q(u_0, u_1, u_2, u_3, u_1', u_2', u_3') := \sum_{i=1}^{3} \mathrm{s}_{(x_0, x_i')}(u_0, u_i') + \sum_{i=1}^{3} \mathrm{s}_{(x_i, x_i')}(u_i, u_i')$$
$$+ \mathrm{s}_{(x_1', x_2')}(u_1', u_2') + \mathrm{s}_{(x_2', x_3')}(u_2', u_3') + \mathrm{s}_{(x_3', x_1')}(u_3', u_1'),$$

and the quadratic form $\widetilde{q}$ on $(\mathbb{R}^3)^4$ by setting:

$$\widetilde{q}(u_0, u_1, u_2, u_3) := \inf \left\{ q(u_0, u_1, u_2, u_3, u_1', u_2', u_3');\ (u_1',\ u_2', u_3') \in (\mathbb{R}^3)^3 \right\}.$$

Let us compute the kernel K of $\widetilde{q}$: (u_0, u_1, u_2, u_3) belongs to K if and only if there exist (u_1', u_2', u_3') such that

$$\mathrm{s}_{(x_1', x_2')}(u_1', u_2') = 0, \quad \mathrm{s}_{(x_2', x_3')}(u_2', u_3') = 0, \quad \mathrm{s}_{(x_3', x_1')}(u_3', u_1') = 0$$

and for any $i =$ 1,2 or 3,

$$\mathrm{s}_{(x_0, x_i')}(u_0, u_i') = 0, \quad \mathrm{s}_{(x_i, x_i')}(u_i, u_i') = 0.$$

As the points (x_0, x_1', x_2', x_3') are not coplanar, these conditions are equivalent to the existence of a vector w such that, for any $i = 1$, 2 or 3,

$$u_i' - u_0 = w \wedge (x_i' - x_0) \quad \text{and} \quad f_i \cdot (u_i - (u_0 + w \wedge (x_i - x_0))) = 0.$$

This implies that (u_0, u_1, u_2, u_3) belongs to K if and only if there exists $w \in \mathbb{R}^3$ such that

$$w \cdot ((x_i - x_0) \wedge f_i) = f_i \cdot (u_i - u_0), \tag{4.15}$$

for any $i = 1$, 2 or 3. As the family $\{f_1, f_2, f_3, x_1 - x_0, x_2 - x_0, x_3 - x_0\}$ has rank 3 and as the equilibrium property imposes $\sum_{i=1}^{3}((x_i - x_0) \wedge f_i) = 0$, the family of vectors $((x_i - x_0) \wedge f_i)$ has rank 2. The existence of w satisfying (4.15) is equivalent to $\sum_{i=1}^{3} f_i \cdot (u_i - u_0) = 0$. The codimension of the kernel of $\widetilde{q}$ is one: there exists $\lambda > 0$ such that

$$\widetilde{q}(u_0, u_1, u_2, u_3) = \lambda \left(\sum_{i=1}^{3} f_i \cdot (u_i - u_0) \right)^2 = \lambda \left(\sum_{i=0}^{3} f_i \cdot u_i \right)^2.$$

[2] Indeed, if the quantity $det(x_1' - x_0, x_2' - x_0, x_3' - x_0)$ which is a polynomial function of ϵ with degree three vanishes for more than three values of ϵ, then all its coefficients vanish. In particular, at order zero we find that the points (x_0, x_1, x_2, x_3) are coplanar. Then, considering the coefficient of order 1 and taking into account the balance equation, some algebra shows that the rank of the family $\{x_1 - x_0, x_2 - x_0, x_3 - x_0, f_1, f_2, f_3\}$ would be smaller than three.

Owing to Theorem 2 and Property 3, for any $g \in L^\infty(\mathbb{R}^3, \mathbb{R}^+)$, the functional $D_{q,(x_0,x_i,x_i'),g}$ belongs to $\overline{\mathfrak{E}_\nu}$. Owing to Theorem 4, the functional $D_{\widetilde{q},(x_i),g}$ belongs also to $\overline{\mathfrak{E}_\nu}$. As $L_{(f_i),(x_i),g}$ coincides with $\lambda^{-1} D_{\widetilde{q},(x_i),g}$, the lemma is proved in the non-planar case.

Let us now consider the general case: for any family of four distinct points (x_i), it is easy to check that there exists a family (g_i) such that the system of forces (g_i, x_i) is balanced and such that the family $\{g_1, g_2, g_3, x_1 - x_0, x_2 - x_0, x_3 - x_0\}$ has rank 3.

Then, for any $\eta > 0$ small enough, the family of vectors $f_i + \eta g_i$ and $x_i - x_0$ has rank 3.[3]. Using the previous result, we know that the functional $L_{(f_i+\eta g_i),(x_i),g}$ belongs to the closure $\overline{\mathfrak{E}_\nu}$. Let η tend to 0, clearly $L_{(f_i+\eta g_i),(x_i),g}$ τ-converges to $L_{(f_i),(x_i),g}$. Therefore, owing to Property 2, the functional $L_{(f_i),(x_i),g}$ belongs also to $\overline{\mathfrak{E}_\nu}$. □

Remark 6 *As $\mathfrak{L}_3 \subset \mathfrak{L}_4$, this proves also that $\mathfrak{L}_3 \subset \overline{\mathfrak{E}_\nu}$.*

Theorem 5 *For any integer $p > 1$, we have $\mathfrak{L}_p \subset \overline{\mathfrak{E}_\nu}$.*

Proof: We use an induction argument with respect to p. The case $p = 2$ has already been considered in Theorem 2. The case $p \leq 4$ has been considered in Lemma 5 and Remark 6.

Let $p > 4$ and assume $\mathfrak{L}_t \subset \overline{\mathfrak{E}_\nu}$ for any $t < p$. Consider an element $L_{(f_i),(x_i),g}$ of $\mathfrak{L}_p$. Let r be the integer part of $p/2$ and $y \in \Omega \setminus \{x_1, \dots x_p\}$ such that $y - x_1$ is orthogonal to $\sum_{i=2}^{r+1} (x_i - x_1) \wedge f_i$. Then there exists f and f' in $\mathbb{R}^3$ such that both families $((f, y), (f_1 + f', x_1), (f_2, x_2), \dots, (f_{r+1}, x_{r+1}))$ and $((-f, y), (-f', x_1), (f_{r+2}, x_{r+2}), (f_{r+3}, x_{r+3}), \dots, (f_p, x_p))$ are balanced systems of forces. Let us define the quadratic forms q' and q'' respectively on $(\mathbb{R}^3)^{r+2}$ and $(\mathbb{R}^3)^{p-r+1}$ by

$$q'(v, u_1, \dots, u_{r+1}) := \left(f \cdot v + f' \cdot u_1 + \sum_{i=1}^{r+1} f_i \cdot u_i \right)^2,$$

$$q''(v, u_1, u_{r+2}, \dots, u_p) := \left(-f \cdot v - f' \cdot u_1 + \sum_{i=r+2}^{p} f_i \cdot u_i \right)^2.$$

The functionals $D_{q',(y,x_1,\dots,x_{r+1}),g}$ and $D_{q'',(y,x_1,x_{r+2},\dots,x_p),g}$ belong respectively to $\mathfrak{L}_{r+2}$ and $\mathfrak{L}_{p-r+1}$. As, when $p > 4$, $r + 2 < p$ and $p - r + 1 < p$,

[3] Indeed, if the rank of the family $\{x_1 - x_0, i = 1, 2, 3\} \cup \{f_i + \eta g_i, i = 1, 2, 3\}$ is smaller than three for more than three values of η, then, for any $(i, j, k) \in \{1, 2, 3\}^3$, the quantity $[(x_i - x_0) \wedge (f_j + \eta g_j)] \wedge [(x_k - x_0) \wedge (f_j + \eta g_j)]$ (which is a polynomial function of η with degree 2) should vanish. In particular, taking only into account the higher order terms we should have

$$[(x_i - x_0) \wedge g_j] \wedge [(x_k - x_0) \wedge g_j] = 0,$$

and the rank of the family $\{x_1 - x_0, i = 1, 2, 3\} \cup \{g_i, i = 1, 2, 3\}$ would be smaller than three.

the induction assumption implies that they belong to $\overline{\mathfrak{E}_\nu}$ and so does their sum.

Now let us compute the infimum

$$\widetilde{q}(u_1, u_2, \ldots, u_p) := \inf_{v \in \mathbb{R}^3} \{2q'(v, u_1, \ldots, u_{r+1}) + 2q''(v, u_1, u_{r+2}, \ldots, u_p)\}.$$

As this infimum is achieved when

$$f \cdot v = \frac{1}{2}\left(-\left(f' \cdot u_1 + \sum_{i=1}^{r+1} f_i \cdot u_i\right) + \left(-f' \cdot u_1 + \sum_{i=r+2}^{p} f_i \cdot u_i\right)\right),$$

we get

$$\widetilde{q}(u_1, u_2, \ldots, u_p) = \left(\sum_{i=1}^{p} f_i \cdot u_i\right)^2.$$

Thus $L_{(f_i),(x_i),g}$ coincides with $D_{\widetilde{q},(x_i),g}$. Then, owing to Theorem 4, the functional $L_{(f_i),(x_i),g}$ belongs to $\overline{\mathfrak{E}_\nu}$. The assertion $\mathfrak{L}_t \subset \overline{\mathfrak{E}_\nu}$ remains valid for $t = p$. □

Theorem 6 *For any $p > 1$, we have $\mathfrak{Q}_p \subset \overline{\mathfrak{E}_\nu}$.*

Proof: Let $D_{q,(x_i),g}$ be in $\mathfrak{Q}_p$. The non-negative quadratic form q on $(\mathbb{R}^3)^p$ can be considered as a quadratic form on $\mathbb{R}^{3p}$ and diagonalized in an orthogonal basis. It is the sum of the square of $3p$ linear forms on $\mathbb{R}^{3p}$ (that is on $(\mathbb{R}^3)^p$) which vanish on the kernel of q. Hence, there exist $3p$ families of p vectors (f_i^j) $(i = 1, \ldots p,\ j = 1, \ldots 3p)$ such that

$$q(u_1, \ldots, u_p) = \sum_{j=1}^{3p}\left(\sum_{i=1}^{p}\left(f_i^j \cdot u_i\right)^2\right).$$

Moreover, for any j, the system of forces (f_i^j, x_i) is balanced, since $(q, (x_i))$ is objective: consequently each functional $L_{(f_i^j),(x_i),g}$ belongs to $\mathfrak{L}_p$ and then, owing to Theorem 5, it belongs to $\overline{\mathfrak{E}_\nu}$. Using Remark 5, we get that their sum $D_{q,(x_i),g} = \sum_{j=1}^{3p} L_{(f_i^j),(x_i),g}$ belongs to $\overline{\mathfrak{E}_\nu}$. □

5. Discretization of a continuous non-local interaction

Theorem 7 *We have $\mathfrak{C} \subset \overline{\mathfrak{E}_\nu}$.*

Proof: For any F in $\mathfrak{C}$, we construct a sequence in $\cup_p \mathfrak{Q}_p$ which τ-converges to F.

Let C^n be the cube $C^n := \frac{1}{n}(-\frac{1}{2}, \frac{1}{2})^3$ and $(C_i^n)_{i=1}^{p_n}$ be the family of all cubes $C_i^n := c_i^n + C^n$ included in Ω and such that $nc_i^n \in \mathbb{Z}^3$. Any compact set included in Ω is, for n large enough, covered almost everywhere by the family (C_i^n) but this is not the case, in general, for the whole domain Ω. So we enlarge some sets C_i^n by defining the family

$$K_i^n := \Big\{ x \in \Omega \ ; \ \|x - c_i^n\| < \|x - c_j^n\|, \ \forall j = 1, \dots, p_n \ , j \neq i \Big\}. \tag{5.1}$$

Note that, for all $i \in \{1, \dots, p_n\}$, $C_i^n \subset K_i^n$ and $\Omega \subset \overline{\cup_{i=1}^{p_n} K_i^n}$. Moreover, since Ω is assumed to be regular, the diameter of the sets K_i^n tends to zero as n tends to infinity $(\max_{i \in \{1,\dots,p_n\}}(\mathrm{diam}(K_i^n)) \to 0)$. Let us introduce the barycenter $c^n := (p_n)^{-1} \sum_{i=1}^{p_n} c_i^n$ of the centers (c_i^n) and J^n their central inertial matrix which is defined by

$$J^n \cdot w := \frac{1}{p_n} \sum_{i=1}^{p_n} ((c_i^n - c^n) \wedge w) \wedge (c_i^n - c^n), \quad \forall w \in \mathbb{R}^3. \tag{5.2}$$

For n large enough, the matrix J^n is symmetric positive definite and therefore invertible. Then we can introduce a linear operator Γ^n which associates to any family (u_i^n) in $(\mathbb{R}^3)^{p_n}$, a function $\Gamma^n((u_i^n)) \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$:

$$\Gamma^n\Big((u_i^n)\Big)(x) := \sum_{i=1}^{p_n} \left[u_i^n + \left((J^n)^{-1} \cdot \frac{1}{p_n} \sum_{j=1}^{p_n} (c_j^n - c^n) \wedge u_j^n \right) \wedge (x - c_i^n) \right] \mathbf{1}_{K_i^n}(x). \tag{5.3}$$

The interest of this operator lies in the following properties:

i) $\Gamma^n(w, w, \dots, w)(x) = w$ and $\Gamma^n(w \wedge c_1^n, w \wedge c_2^n, \dots, w \wedge c_{p_n}^n)(x) = w \wedge x$, for almost every x in Ω,

ii) $\|\Gamma^n\big((u_i^n)\big)\|_\infty \leq 2 \max_{i \in \{1,\dots,p_n\}} \|u_i^n\|$,

iii) if a sequence of functions (u^n) converges to u weakly in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$, then so does the sequence $\Gamma^n\Big((⨍_{C_i^n} u^n)\Big)$,

iv) if u belongs to $C^\infty(\Omega, \mathbb{R}^3)$, then the sequence $\Gamma^n\Big(\big(u(c_i^n)\big)\Big)$ converges to u strongly in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$.

Straightforward computations lead to Properties (i). To prove (ii) and (iii), note that the matrix J^n converges to the central inertial matrix of Ω which is positive definite. Therefore $\big\|(J^n)^{-1}\big\|$ is bounded by some constant and so is the quantity $\Big\|(J^n)^{-1} \cdot \frac{1}{p_n} \sum_{j=1}^{p_n} (c_j^n - c^n) \wedge u_j^n\Big\|$. As the diameter of K_i^n tends to zero, the right-hand term in the definition of $\Gamma^n\big((u_i^n)\big)$ tends uniformly to zero. Property (iii) results from the weak convergence of $\sum_{i=1}^{p_n} (⨍_{C_i^n} u^n) \mathbf{1}_{K_i^n}$ to u. Property (iv) is obvious. Indeed the considered convergence is actually uniform.

Let us denote by g_n the function $g_n := \frac{1}{|C^n|} \mathbf{1}_{C^n}$ and q_n the quadratic form on $(\mathbb{R}^3)^{p_n}$ defined by

$$q_n(u_1^n, u_2^n, \dots, u_{p_n}^n) := F\Big(\Gamma^n\big((u_i^n)\big)\Big). \tag{5.4}$$

Properties (i) of $\varGamma^n$ ensure the objectivity of $(q_n,(c_i^n))$: the functional $D_{q_n,(c_i^n),g_n}$ is contained in $\mathfrak{Q}_{p_n}$ and, owing to Theorem 6, in $\overline{\mathfrak{E}_\nu}$.

Let (u^n) be a sequence converging to u weakly in $\mathrm{L}^2(\varOmega,\mathbb{R}^3)$. By convexity, we have

$$\begin{aligned} D_{q_n,(c_i^n),g_n}(u^n) &= \fint_{C^n} q_n(u^n(c_1^n+x),u^n(c_2^n+x),\ldots,u^n(c_{p_n}^n+x))\,dx \\ &\geq q_n\left(\fint_{C_1^n} u^n, \fint_{C_2^n} u^n,\ldots,\fint_{C_{p_n}^n} u^n\right) \\ &\geq F\left(\varGamma^n\Big((\fint_{C_i^n} u^n)\Big)\right). \end{aligned} \tag{5.5}$$

Owing to Property (iii) of $\varGamma^n$ and to the lower-semicontinuity of F, we can pass to the limit and get the lower-bound inequality

$$\liminf_{n\to\infty} D_{q_n,(c_i^n),g_n}(u^n) \geq \liminf_{n\to\infty} F\left(\varGamma\Big((\fint_{C_i^n} u^n)\Big)\right) \geq F(u). \tag{5.6}$$

Now, let u be in $\mathrm{L}^2(\varOmega,\mathbb{R}^3)$. By a density argument we can assume that u belongs to $C^\infty(\varOmega,\mathbb{R}^3)$. As, for any $i\in\{1,\ldots,p_n\}$ and any $x\in C^n$, $\|u(c_i^n+x)-u(c_i^n)\|\leq\frac{1}{n}\|\nabla u\|_\infty$, Property (ii) of $\varGamma^n$ implies that $\|\varGamma^n((u(c_i^n+x)))-\varGamma^n((u(c_i^n)))\|_\infty\leq\frac{2}{n}\|\nabla u\|_\infty$. The continuous quadratic functional F is Lipschitz on every bounded subset of $\mathrm{L}^2(\varOmega,\mathbb{R}^3)$: for some constant C', we obtain, for all $x\in C^n$,

$$\left|q_n(u(c_1^n+x),u(c_2^n+x),\ldots,u(c_{p_n}^n+x)) - q_n(u(c_1^n),u(c_2^n),\ldots,u(c_{p_n}^n))\right| \leq \frac{C'}{n}.$$

Hence

$$\begin{aligned} D_{q_n,(c_i^n),g_n}(u) &= \fint_{C^n} q_n(u(c_1^n+x),u(c_2^n+x),\ldots,u(c_{p_n}^n+x))\,dx \\ &\leq q_n(u(c_1^n),u(c_2^n),\ldots,u(c_{p_n}^n)) + \frac{C'}{n} \\ &\leq F\Big(\varGamma^n\Big(\big(u(c_i^n)\big)\Big)\Big) + \frac{C'}{n}. \end{aligned}$$

Owing to Property (iv) of $\varGamma^n$ and to the continuity of F, we can pass to the limit and get the upper-bound inequality

$$\limsup_{n\to\infty} D_{q_n,(c_i^n),g_n}(u) \leq F(u). \tag{5.7}$$

This inequality, together with the lower-bound inequality (5.6) proves that $D_{q_n,(c_i^n),g_n}$ τ-converges to F. The proof of Theorem 7 is concluded using Property 2. □

6. Moreau-Yosida approximation

The Moreau-Yosida approximation is a standard way to approximate a functional by a continuous one. Restricting it to objective forms does not present any difficulty.

Theorem 8 *We have* $\mathfrak{Q} \subset \overline{\mathfrak{C}}$.

Let $F \in \mathfrak{Q}$ be a lower semi-continuous objective quadratic form and n an positive integer. The Moreau-Yosida approximation of index n of F [12] is the functional defined on $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ by

$$\mathrm{Y}_n(F)(u) = \inf_{v \in \mathrm{L}^2(\Omega,\mathbb{R}^3)} \{F(v) + n\|u - v\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^3)}\}. \tag{6.1}$$

Clearly $\mathrm{Y}_n(F)$ is locally Lipschitz and then continuous on $\mathrm{L}^2(\Omega, \mathbb{R}^3)$. On the other hand, as F is objective, so is its Moreau-Yosida approximation $\mathrm{Y}_n(F)$: $\mathrm{Y}_n(F) \in \mathfrak{C}$.

We have, for any $n_1 < n_2$, and any $u \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$,

$$\mathrm{Y}_{n_1}(F)(u) \leq \mathrm{Y}_{n_2}(F)(u) \leq F(u). \tag{6.2}$$

This implies $\lim Y_n(F)(u) \leq F(u)$. Let us prove the opposite inequality, assuming that $\lim Y_n(F)(u) < +\infty$. For any $n \in \mathbb{N}^*$, there exists v_n such that

$$F(v_n) + n\|u - v_n\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} \leq \mathrm{Y}_n(F)(u) + \frac{1}{n}\ . \tag{6.3}$$

As the right hand side of this inequality is bounded, the sequence (v_n) tends to u. Using the lower semi-continuity of F and passing to the limit in (6.3) we get

$$\lim \mathrm{Y}_n(F)(u) \geq \lim F(v_n) = F(u)\ .$$

Therefore, the pointwise convergence of $\mathrm{Y}_n(F)$ to F is assured. Property 4 assures that the non-decreasing sequence $\mathrm{Y}_n(F)$ τ-converges to F. □

Owing to Theorem 7, F is the τ-limit of a sequence contained in $\overline{\mathfrak{C}_\nu}$. Property 2 assures that F belongs to $\overline{\mathfrak{C}_\nu}$.

This finally concludes the proof of Theorem 1.

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