# Energies with Respect to a Measure and Applications to Low Dimensional Structures


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**Abstract.** We consider functionals of the form

$$F(u) = \int f(x, Du)\, d\mu \qquad \big(u \in \mathcal{D}(\mathbf{R}^n)\big)$$

where $\mu$ is a finite Borel measure on $\mathbf{R}^n$, and we characterize their relaxation $\overline{F}$ with respect to the weak convergence in a suitable Sobolev space $W^{1,p}_\mu$. Applications to low dimensional structures and junctions are given.


## 1. Introduction

The problem of finding variational models for low dimensional elastic structures has been widely considered in the literature (see, for instance, the classical books of Landau and Lifchitz [10], and Love [12]). More recently, the justification of some classical models has been obtained via $\Gamma$-convergence (see, for instance, Acerbi, Buttazzo and Percivale [1], [2], Percivale [15]), or via asymptotic development of solutions (see, for instance, Ciarlet [7], Ciarlet and Destuynder [8], Le Dret and Raoult [11]); this method consists in fattening the structure $S$ into a $n$-dimensional ($n = 3$ in the applications) one, $S_\varepsilon$, having $\varepsilon$ as a thickness parameter, and then in passing to the limit as $\varepsilon \to 0$, obtaining in this way the description of the limit problem.

Here we adopt another, more intrinsic, point of view which consists in describing the structure by means of a measure $\mu$ on $\mathbf{R}^n$; the energy functional will be initially defined by

$$F(u) = \int f(x, Du)\, d\mu \qquad \big(u \in C^1(\mathbf{R}^n; \mathbf{R}^m)\big)$$

where $f$ is the $n$-dimensional energy density integrand. The low dimensional elastic model will be simply obtained by a suitable relaxation $\overline{F}$ of the functional $F$; this will give

$$\overline{F}(u) = \int f_\mu(x, D_\mu u)\, d\mu \qquad \big(u \in W^{1,p}_\mu(\mathbf{R}^n; \mathbf{R}^m)\big)$$

where $W^{1,p}_\mu$ is the space of functions with finite energy, $f_\mu$ is the relaxed integrand, and $D_\mu$ stands for the "tangential gradient" operator with respect to $\mu$. Given a low dimensional manifold $S$ of dimension $k$, it will suffice to take $\mu = H^k \llcorner S$ to obtain the desired elasticity model on $S$.

In Section 2 we introduce the concept of tangential gradient with respect to a measure $\mu$ which enables us to construct the associated Sobolev space $W^{1,p}_\mu$, in Section 3 we state and prove the relaxation result, and in Section 4 we consider some examples of measures which describe junctions of elastic materials with different dimensions. Also, the occurrence of nonlocal effects is pointed out, with possible links to the theory of Dirichlet forms (see Mosco [13],[14]).

## 2. Notation and Preliminary Results

Let $n$ be a positive integer, let $p \in ]1, +\infty[$, and let $\mu$ be a nonnegative finite Borel measure on $\mathbf{R}^n$. As usual, we denote by $p'$ the conjugate exponent of $p$ given by $1/p + 1/p' = 1$. We consider the space

$$X^{p'}_\mu = \big\{\phi \in L^{p'}_\mu(\mathbf{R}^n;\mathbf{R}^n) \ : \ \mathrm{div}(\phi\mu) \in L^{p'}_\mu(\mathbf{R}^n)\big\},$$

where the divergence operator div is intended in the sense of distributions on $\mathbf{R}^n$. In other words, a function $\phi$ belongs to $X^{p'}_\mu$ if and only if $\phi \in L^{p'}_\mu(\mathbf{R}^n;\mathbf{R}^n)$ and there exists $K > 0$ such that

$$\int \phi \cdot D\psi \, d\mu \leq K\|\psi\|_{L^p_\mu} \qquad \forall \psi \in \mathcal{D}(\mathbf{R}^n).$$

For $\mu-$a.e. $x \in \mathbf{R}^n$ we define

$$T^p_\mu(x) = \mu - ess\bigcup\{\phi(x) \ : \ \phi \in X^{p'}_\mu\}$$

where the $\mu-$essential union is defined as a $\mu-$measurable closed valued multifunction such that (see for instance Bouchitté and Valadier [4]):

- $\phi \in X^{p'}_\mu \ \Rightarrow \ \phi(x) \in T^p_\mu(x)$ $\mu-$a.e.,
- $T^p_\mu(x) \subset \Gamma(x)$ $\mu-$ a.e. for all multifunctions $\Gamma(x)$ satisfying the properties above,
- there exists a sequence $(\phi_h)$ in $X^{p'}_\mu$ such that $T^p_\mu(x) = cl(\{\phi_h(x) \ : \ h \in \mathbf{N}\})$ $\mu-$a.e..

It is immediate to see that for $\mu-$a.e. $x \in \mathbf{R}^n$ the set $T^p_\mu(x)$ is a linear subspace of $\mathbf{R}^n$ that we call the *tangent space* of $\mu$ at $x$. We denote by $P_\mu(x,\cdot)$ the orthogonal projection on $T^p_\mu(x)$.

Given $u \in \mathcal{D}(\mathbf{R}^n)$ we define, for $\mu-$a.e. $x \in \mathbf{R}^n$, the tangential derivative $D_\mu u(x)$ as the projection of $Du(x)$ on $T^p_\mu(x)$:

$$D_\mu u(x) = P_\mu\big(x, Du(x)\big).$$

We remark that $D_\mu u(x)$, as an element of $(L^p_\mu)^n$, depends only on the equivalence class of $u$ in the space $L^p_\mu$. Then we can consider the operator on $L^p_\mu$ given by $u \mapsto D_\mu u$ and whose domain is $\mathcal{D}(\mathbf{R}^n)$; to extend this operator to a larger domain we need the following closability result.

**Proposition 2.1.** *Let $(u_h)$ in $\mathcal{D}(\mathbf{R}^n)$ and $v$ in $(L^p_\mu)^n$ be such that*

$$u_h \rightharpoonup 0, \quad D_\mu u_h \rightharpoonup v \quad \textit{weakly in } L^p_\mu.$$

*Then $v = 0$ $\mu-$a.e..*

Before proving this result, we remark that it enables us to introduce the smallest closed extension of $D_\mu$ (still denoted by $\mathcal{D}_\mu$) by setting

$$w = D_\mu u \quad \iff \quad \exists u_h \in \mathcal{D}(\mathbf{R}^n) \ : \ (u_h, D_\mu u_h) \to (u, w) \text{ in } (L^p_\mu)^{n+1}.$$

We define the Sobolev space $W^{1,p}_\mu$ as the domain of the extension above, endowed with the norm

$$\|u\|_{1,p,\mu} = \|u\|_{L^p_\mu} + \|D_\mu u\|_{L^p_\mu}.$$

We notice that $W^{1,p}_\mu$ is the completion of $\mathcal{D}(\mathbf{R}^n)$ with respect to the norm $\|\cdot\|_{1,p,\mu}$. On this separable Banach space the weak convergence is defined by

$$u_h \rightharpoonup u \text{ weakly in } W^{1,p}_\mu \qquad \Longleftrightarrow \qquad \begin{cases} u_h \rightharpoonup u \text{ weakly in } L^p_\mu \\ D_\mu u_h \rightharpoonup D_\mu u \text{ weakly in } L^p_\mu. \end{cases}$$

Using Proposition 2.1, it can be easily checked that for every $p \in ]1,+\infty[$ the space $W^{1,p}_\mu$ is reflexive. In order to describe the formula of integration by parts on $W^{1,p}_\mu$, we notice that if $\phi \in X^{p'}_\mu$ then $\phi(x) \in T^p_\mu(x)$ for $\mu-$a.e. $x$, and so, for every $u \in \mathcal{D}(\mathbf{R}^n)$,

$$\begin{aligned}\int D_\mu u \cdot \phi \, d\mu &= \int P_\mu(x, Du) \cdot \phi \, d\mu \\ &= \int Du \cdot \phi \, d\mu = -\langle u, \operatorname{div}(\phi\mu)\rangle.\end{aligned}$$

which, by means of a density argument, yields

$$\int D_\mu u \cdot \phi \, d\mu = -\langle u, \operatorname{div}(\phi\mu)\rangle \qquad \text{for every } u \in W^{1,p}_\mu. \tag{2.1}$$

**Proof of Proposition 2.1.** Using the integration by parts formula (2.1) for every $\phi \in X^{p'}_\mu$, we obtain

$$\int v \cdot \phi \, d\mu = \lim_{h\to+\infty} \int D_\mu u_h \cdot \phi \, d\mu = -\lim_{h\to+\infty} \int u_h \operatorname{div}(\phi\mu) \, d\mu = 0.$$

We notice that $X^{p'}_\mu$ verifies the following locality property:

$$\phi \in X^{p'}_\mu, \quad \varphi \in \mathcal{D}(\mathbf{R}^n) \quad \Rightarrow \quad \phi\varphi \in X^{p'}_\mu.$$

Hence, by using an argument of commutation between $\int$ and $\sup$ (see Lemma 4.3 of Bouchitté and Dal Maso [3], or Bouchitté and Valadier [4]), we get

$$0 = \sup_{\phi \in X^{p'}_\mu} \int v \cdot \phi \, d\mu = \int \mu - ess \sup_{\phi \in X^{p'}_\mu} (v \cdot \phi) \, d\mu.$$

We can choose a sequence $\phi_h$ in $X^{p'}_\mu$ such that $\mu-\operatorname{ess\,sup}_{\phi \in X^{p'}_\mu}(v \cdot \phi)(x) = \sup_h (v \cdot \phi_h)(x)$ and such that the set $\{\phi_h(x) \ : \ h \in \mathbf{N}\}$ is dense in $T^p_\mu(x)$, $\mu-$a.e. We deduce

$$0 = \int \sup_h (v \cdot \phi_h) \, d\mu = \int \sup_{z \in T^p_\mu(x)} (v(x) \cdot z) \, d\mu$$

which implies that

$$v(x) \in \big(T^p_\mu(x)\big)^\perp \qquad \text{for } \mu\text{-a.e. } x. \tag{2.2}$$

Since the linear space $X = \big\{v \in (L^p_\mu)^n \ : \ v(x) \in T^p_\mu(x) \text{ for } \mu\text{-a.e. } x\big\}$ is closed, and since $D_\mu u_h \in X$, we have $v \in X$. This, together with (2.2), implies that $v = 0$ $\mu-$ a.e. ■

**Remark 2.2.** Here we list some properties of the tangent space $T^p_\mu(x)$, recalling that $T^p_\mu(x)$ is only $\mu-$a.e. defined. Proofs are straightforward and left to the reader.

(i) For every open subset $A \subset \mathbf{R}^n$, the following locality property holds:

$$\mu \llcorner A = \nu \llcorner A \quad \Rightarrow \quad T^p_\mu(x) = T^p_\nu(x) \quad \mu - \text{a.e. on } A.$$

(ii) If $p < q$, then $T^p_\mu(x) \subset T^q_\mu(x)$. For all the measures $\mu$ we considered (see Examples 2.3–2.5) we find that the space $T^p_\mu(x)$ does not depend on $p$. We think that, in general, the inclusion could be strict, although we were not able to find any counterexample.

(iii) If $\nu$ is absolutely continuous with respect to $\mu$, then $T^p_\nu(x) \subset T^p_\mu(x)$ for $\nu-$a.e. $x$.

(iv) If $\mu_1$, $\mu_2$ are Radon measures on $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively, then the tensor product $\mu = \mu_1 \otimes \mu_2$ satisfies

$$T^p_\mu(x,y) = T^p_{\mu_1}(x) \times T^p_{\mu_2}(y) \qquad \text{for } \mu\text{−a.e. } (x,y) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}.$$

Some other useful properties of the tangent space $T^p_\mu(x)$ are included in the Appendix (Lemma 5.2 and Corollaries 5.4, 5.5).

In the following examples we denote by $1_E$ the characteristic function of $E$ defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise.

**Example 2.3.** Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ with a Lipschitz boundary, and let $\mu = H^n \llcorner \Omega$. Then we get

$$\begin{aligned}
&X^{p'}_\mu = \big\{\phi \in L^{p'}(\Omega;\mathbf{R}^n) \ :\ \operatorname{div}\phi \in L^{p'}(\Omega),\ \phi\cdot\nu = 0 \text{ on } \partial\Omega\big\},\\
&T^p_\mu(x) = \mathbf{R}^n \quad \text{for a.e. } x \in \Omega,\\
&D_\mu u = 1_\Omega Du,\\
&W^{1,p}_\mu = W^{1,p}(\Omega).
\end{aligned}$$

**Example 2.4.** Let $S$ be a smooth compact manifold in $\mathbf{R}^n$, of dimension $k < n$, with smooth boundary $\partial S$, and let $\mu = H^k \llcorner S$. Then (see Appendix, Corollary 5.4)

$$\begin{aligned}
&X^{p'}_\mu = \big\{\phi \in L^{p'}(S;\mathbf{R}^n) \ :\ \operatorname{div}_S \phi_S \in L^{p'}(S),\ \phi_\nu = 0 \text{ on } S,\ \phi_S\cdot\tau_S = 0 \text{ on } \partial S\big\},\\
&T^p_\mu(x) = T_S(x) \quad \text{(independent of } p),\\
&D_\mu u = 1_S D_S u,\\
&W^{1,p}_\mu = W^{1,p}(S).
\end{aligned}$$

where the subscript $S$ and the subscript $\nu$ denote respectively the tangential and normal components to $S$, and $\tau_S$ is the versor tangential to $S$ and normal to $\partial S$.

Having in mind the problem of junctions between multi-dimensional structures, we can generalize this example to the case where $S$ is a finite union of smooth compact manifolds $S_i$ ($i = 1,\dots,N$). Assume that $S_i$ has dimension $k_i$ and that the corresponding measures $\mu_i = H^{k_i} \llcorner S_i$ are mutually singular. Then, setting $\mu = \sum_i \mu_i$, we can prove (see Appendix, Corollary 5.5) that

$$\begin{aligned}
&T^p_\mu(x) = T_{S_i}(x) \qquad \mu_i - \text{a.e.},\\
&D_\mu u = \sum_i 1_{S_i} D_{S_i} u,\\
&W^{1,p}_\mu = \big\{u \ :\ u_{|S_i} \in W^{1,p}(S_i) \text{ for every } i\big\}.
\end{aligned}$$

**Example 2.5.** Let $C$ be a closed subset of $\mathbf{R}$ such that $\mathbf{R}\setminus C$ is dense in $\mathbf{R}$, let $\alpha \in [0,1]$ be such that $H^\alpha(C) < +\infty$, and let $\mu = H^\alpha \llcorner C$. Then for every $p$, the tangent space $T^p_\mu(x)$ reduces to $\{0\}$, so that $W^{1,p}_\mu = L^p_\mu$ and $D_\mu u = 0$ for every $u \in W^{1,p}_\mu$. Indeed, if $\phi \in X^{p'}_\mu$ we have that, for some suitable $g \in L^{p'}_\mu$,

$$\frac{d}{dx}(\phi\mu) = g\mu. \tag{2.3}$$

As $g\mu$ is a Radon measure, we deduce from (2.3) that the measure $\phi\mu$ can be written as $fH^1$, where $f$ is a function with bounded variation.

In the case $\alpha < 1$, the measure $\phi\mu$ is singular with respect to $H^1$; hence it vanishes and $\phi = 0$ $\mu$−a.e. (in fact the same conclusion holds for all measures $\mu$ which are singular with respect to $H^1$).
In the case $\alpha = 1$, from (2.3) we get that $f = \phi 1_C$ is absolutely continuous. As it vanishes on the dense set $\mathbf{R} \setminus C$, by continuity we have $\phi = 0$ on $C$ and so $\phi = 0$ $\mu$−a.e..

## 3. The Relaxation Result

Given a function $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ such that

(3.1) for every $z \in \mathbf{R}^n$ the function $f(\cdot, z)$ is $\mu$−measurable,

(3.2) for $\mu$−a.e. $x \in \mathbf{R}^n$ the function $f(x, \cdot)$ is convex,

(3.3) there exist $c_1 > 0$, $c_2 > 0$ such that

$$c_1|z|^p \le f(x,z) \le c_2\big(1 + |z|^p\big) \qquad \forall (x,z) \in \mathbf{R}^n \times \mathbf{R}^n,$$

we consider the associated functional defined on $W^{1,p}_\mu$ by

$$F(u) = \begin{cases} \int f(x, Du)\,d\mu & \text{if } u \in \mathcal{D}(\mathbf{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

Our goal is to represent the relaxed functional

$$\overline{F} = \max\big\{G : W^{1,p}_\mu \to \mathbf{R} \ : \ G \text{ is weakly l.s.c.}, G \le F\big\}$$

in a suitable integral form. To this aim, we introduce the function $f_\mu : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by

$$f_\mu(x,z) = \inf\big\{f(x, z+\xi) \ : \ \xi \in \big(T^p_\mu(x)\big)^\perp\big\}.$$

We are now in a position to state our relaxation result.

**Theorem 3.1.** *For every $u \in W^{1,p}_\mu$ we have*

$$\overline{F}(u) = \int f_\mu(x, D_\mu u)\,d\mu.$$

**Proof.** By assumption (3.3) and by the reflexivity of $W^{1,p}_\mu$, the relaxed functional $\overline{F}$ does not change if we substitute the weak $W^{1,p}_\mu$−convergence by the weak $L^p_\mu$−convergence, and since $F$ is convex, this last one can be substituted by the strong $L^p_\mu$−convergence. Then, a well known result of convex analysis (see for instance [6]) states that $\overline{F}$ is the bipolar functional of $F$ in the duality $\langle L^p_\mu, L^{p'}_\mu\rangle$:

$$\overline{F}(u) = F^{**}(u) = \sup\big\{\langle u, v\rangle - F^*(v) \ : \ v \in L^{p'}_\mu\big\} \tag{3.4}$$

where

$$F^*(v) = \sup\big\{\langle w, v\rangle - F(w) \ : \ w \in L^p_\mu\big\}.$$

Let us consider the densily defined linear operator $A$ from $L^p_\mu$ to $(L^p_\mu)^n$ given by $A(u) = Du$ with domain $\mathcal{D}(\mathbf{R}^n)$, and let $A^*$ be its adjoint operator. For $\phi \in D(A^*)$ and $\psi \in \mathcal{D}(\mathbf{R}^n)$ we have

$$\langle A^*\phi, \psi\rangle = \langle \phi, A\psi\rangle = \int \phi \cdot D\psi\,d\mu,$$

so that $D(A^*) = X^{p'}_\mu$ and, using (2.1), we get $\mathcal{A}^*\phi = -\operatorname{div}(\phi\mu)$.

Set for every $w \in L^p_\mu$

$$I_f(w) = \int f\big(x, w(x)\big)\, d\mu;$$

by (3.3) the functional $I_f$ is convex and continuous on $L^p_\mu$. Its polar functional $I^*_f$ (see [6]) is given by $I^*_f(\phi) = \int f^*\big(x, \phi(x)\big)\, d\mu$. Then, by Theorem 5.1 of the Appendix, the polar of $F(u) = I_f(Au)$ is given by

$$F^*(v) = \inf\Big\{\int f^*(x, \phi)\, d\mu \;:\; A^*\phi = v\Big\}.$$

Thus equation (3.4) becomes

$$\begin{aligned}\overline{F}(u) &= \sup\Big\{\langle u, v\rangle - \int f^*\big(x, \phi(x)\big)\, d\mu \;:\; v \in L^{p'}_\mu,\ \phi \in X^{p'}_\mu,\ A^*\phi = v\Big\} = \\ &= \sup\Big\{ -\int u \operatorname{div}(\phi\mu)\, d\mu - \int f^*(x, \phi)\, d\mu \;:\; \phi \in X^{p'}_\mu\Big\} = \\ &= \sup\Big\{\int \big(\phi \cdot D_\mu u - f^*(x, \phi)\big)\, d\mu \;:\; \phi \in X^{p'}_\mu\Big\}.\end{aligned}$$

Using the localization property

$$\theta \in \mathcal{D}(\mathbf{R}^n), \quad \phi \in X^{p'}_\mu \quad \Rightarrow \quad \theta\phi \in X^{p'}_\mu$$

and the same argument of commutation used in the proof of Proposition 2.1 (see [3], [4]), we get

$$\overline{F}(u) = \int \mu - ess \sup_{\phi \in X^{p'}_\mu} \big(\phi \cdot D_\mu u - f^*(x, \phi)\big)\, d\mu.$$

We can choose a sequence $\phi_h$ such that the set $\{\phi_h(x) \;:\; h \in \mathbf{N}\}$ is dense in $T^p_\mu(x)$ $\mu-$a.e. and such that

$$\overline{F}(u) = \int \sup_h \big(\phi_h \cdot D_\mu u - f^*(x, \phi_h(x))\big)\, d\mu.$$

Then, by the continuity of $f^*$ due to (3.3), we get for every $u \in W^{1,p}_\mu$

$$\overline{F}(u) = \int g(x, D_\mu u)\, d\mu.$$

where

$$g(x, z) = \sup\big\{w \cdot z - f^*(x, w) \;:\; w \in T^p_\mu(x)\big\}.$$

Now, an easy computation shows that

$$\begin{aligned}f^{**}_\mu(x, z) &= \sup_{z^*}\big[z \cdot z^* - \sup_t \big(t \cdot z^* - f_\mu(x, t)\big)\big] = \\ &= \sup_{z^*}\big[z \cdot z^* - \sup\big\{t \cdot z^* - f(x, t + s) \;:\; t \in \mathbf{R}^n,\ s \in \big(T^p_\mu(x)\big)^\perp\big\}\big] = g(x, z).\end{aligned}$$

Since $f_\mu(x, \cdot)$ is convex and $f_\mu \le f$, $f_\mu(x, \cdot)$ is a continuous function. Hence $f^{**}_\mu = f_\mu$ and $g = f_\mu$. ∎

**Remark 3.2.** A result similar to Theorem 3.1 holds for vector valued functions $u : \mathbf{R}^n \to \mathbf{R}^m$. In this case $D_\mu u$ is a $m \times n$ matrix which satisfies the formula

$$\int \phi : D_\mu u\, d\mu = -\int u \cdot \operatorname{div}(\phi\mu)\, d\mu = -\sum_{i,j} \int u_i D_j(\phi_{ij}\mu)\, d\mu$$

for every matrix $\phi(x) = \phi_{ij}(x)$ such that $\phi_{i,\cdot} \in X^{p'}_\mu$ for all $i$. In this case

$$f_\mu(x, z) = \inf\big\{f(x, z + \xi) \;:\; \ker\xi \supset T^p_\mu(x)\big\}.$$

Let now $\Omega$ be a bounded open subset of $\mathbf{R}^n$ with a Lipschitz boundary, and let $\mu$ be a measure vanishing outside $\Omega$ such that $\mu \geq H^n \llcorner \Omega$, so that (see Example 2.3) $W^{1,p}_\mu \subset W^{1,p}(\Omega)$. Taking into account (3.3), we may consider the relaxation problem for $F$ with respect to the weak $W^{1,p}(\Omega)$ convergence, and this will provide a relaxed functional that we denote by $\mathcal{F}$. The relation between $\mathcal{F}$ and $\overline{F}$ is given by the following result.

**Theorem 3.3.** *For every $u \in W^{1,p}(\Omega)$ we have*

$$\mathcal{F}(u) = \inf\big\{\overline{F}(v) \ : \ v \in W^{1,p}_\mu, \ v = u \ H^n\text{-a.e. on } \Omega\big\}. \tag{3.5}$$

**Proof.** Denote by $\mathcal{F}_0$ the right-hand side of (3.5), take $u \in W^{1,p}(\Omega)$, and take a sequence $(u_h)$ of smooth functions such that $u_h \to u$ weakly in $W^{1,p}(\Omega)$ and $F(u_h)$ is bounded. Then, by (3.3) and the reflexivity of $W^{1,p}_\mu$, (a subsequence of) $(u_h)$ converges weakly in $W^{1,p}_\mu$ to some $v \in W^{1,p}_\mu$, and, since $\mu \geq H^n \llcorner \Omega$, we have $v = u$ $H^n$-a.e. on $\Omega$. By using Theorem 3.1 we get

$$\mathcal{F}_0(u) \leq \overline{F}(v) \leq \liminf_{h\to+\infty} F(u_h)$$

so that, by the arbitrariness of $(u_h)$, inequality $\mathcal{F}_0 \leq \mathcal{F}$ is proved.
In order to prove the opposite inequality, for fixed $u \in W^{1,p}(\Omega)$ and $\varepsilon > 0$, take $v \in W^{1,p}_\mu$ with $v = u$ $H^n$−a.e. on $\Omega$ such that $\mathcal{F}_0(u) \geq \overline{F}(v) - \varepsilon$ (this if there exists at least one of such $v$, otherwise the inequality $\mathcal{F}_0(u) \geq \mathcal{F}(u)$ is trivial). By the relaxation result of Theorem 3.1 there exists a sequence $(u_h)$ of smooth functions such that $u_h \to v$ weakly in $W^{1,p}_\mu$ and

$$\overline{F}(v) = \lim_{h\to+\infty} F(u_h).$$

Since $\mu \geq H^n \llcorner \Omega$ the sequence $(u_h)$ converges to $u$ weakly in $W^{1,p}(\Omega)$, so that

$$\mathcal{F}_0(u) \geq \overline{F}(v) - \varepsilon \geq \mathcal{F}(u) - \varepsilon$$

and, by the arbitrariness of $\varepsilon$, we get $\mathcal{F}_0(u) \geq \mathcal{F}(u)$, and the proof is concluded. ■

**Remark 3.4.** A formula analogous to (3.5) holds if $\mu \geq \nu$ and $\mathcal{F}_\nu$ is the relaxed functional of $F$ with respect to the $L^p_\nu$ convergence, that is

$$\mathcal{F}_\nu(u) = \inf\big\{\overline{F}(v) \ : \ v \in W^{1,p}_\mu, \ v = u \ \nu\text{-a.e.}\big\}.$$

In particular, this is true for the measure $\nu = \frac{d\mu}{dx}dx$.

## 4. Some Examples

In this section we present some examples which fall within the framework considered in this paper. We start with an example similar to the one already considered in Buttazzo and Dal Maso [5], Section 6.

**Example 4.1.** Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ ($n \geq 3$), let $x_1$ and $x_2$ be two points in $\Omega$ and $\Gamma$ be a smooth simple path in $\Omega$ joining $x_1$ to $x_2$. Fix $p > 1$, define the function

$$a(x) = |x - x_1|^q + |x - x_2|^q \qquad (q < p - n),$$

let $b : \Gamma \to \mathbf{R}$ be a positive $H^1$-measurable function, set $\mu = a(x)H^n \llcorner \Omega + b(x)H^1 \llcorner \Gamma$, and consider the integrand

$$f(x, z) = |z|^p.$$

As an application of the results of Section 3, we get

$$\overline{F}(v) = \int_\Omega a(x)|Dv|^p\,dx + \int_\Gamma b(x)|D_\tau v|^p\,dH^1 \qquad \forall v \in W^{1,p}_\mu.$$

We compute now the relaxed functional $\mathcal{F}$ of Theorem 3.3. It should be noticed that a function $u \in W^{1,p}(\Omega)$ with $\mathcal{F}(u) < +\infty$ must be defined in $x_1$ and in $x_2$; indeed, the capacity in $\Omega$ of the two points $x_1, x_2$ with respect to the energy $\int_\Omega a(x)|Du|^p\,dx$ is positive. Therefore, the equality $u = v$ in the $\inf$ of formula (3.5) must hold at $x_1$ and at $x_2$, whereas, being $\Gamma$ of capacity zero in $\Omega$, $u$ and $v$ are completely independent on $\Gamma \setminus \{x_1, x_2\}$. Hence, formula (3.5) gives

$$\mathcal{F}(u) = \int_\Omega a(x)|Du|^p\,dx + \inf\left\{\int_\Gamma b(x)|D_\tau v|^p\,dH^1 \ :\ v(x_i) = u(x_i),\ i = 1,2\right\},$$

and, after an easy computation, one obtains

$$\mathcal{F}(u) = \int_\Omega a(x)|Du|^p\,dx + |u(x_2) - u(x_1)|^p \left(\int_\Gamma b(x)^{1/(1-p)}\,dH^1\right)^{1-p}.$$

Similar results hold if the coefficient $a$ "charges" a finite number of points $x_1, \ldots, x_N$ of $\Gamma$, that is

$$a(x) = \sum_{i=1}^N |x - x_i|^q \qquad (q < p - n).$$

For instance, in the case of Figure 4.1 we get

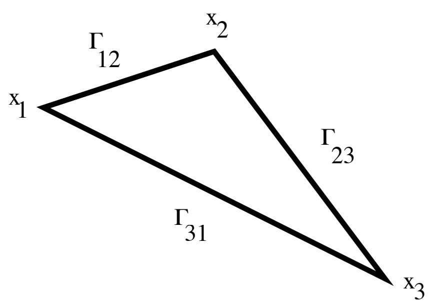


Figure 4.1.

$$\begin{aligned}\mathcal{F}(u) = \int_\Omega a(x)|Du|^p\,dx &+ |u(x_1) - u(x_2)|^p \left(\int_{\Gamma_{12}} b(x)^{1/(1-p)}\,dH^1\right)^{1-p} + \\ &+ |u(x_2) - u(x_3)|^p \left(\int_{\Gamma_{23}} b(x)^{1/(1-p)}\,dH^1\right)^{1-p} + \\ &+ |u(x_3) - u(x_1)|^p \left(\int_{\Gamma_{31}} b(x)^{1/(1-p)}\,dH^1\right)^{1-p},\end{aligned}$$

whereas in the case of Figure 4.2

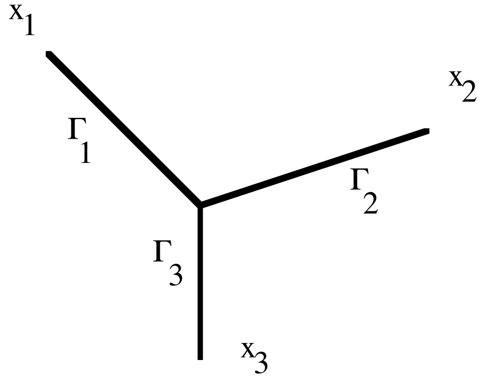


Figure 4.2.

$$\mathcal{F}(u) = \int_\Omega a(x)|Du|^p\,dx + \inf_{s\in\mathbf{R}} \left\{ \sum_{i=1}^{3} |u(x_i) - s|^p \left( \int_{\Gamma_i} b(x)^{1/(1-p)}\,dH^1 \right)^{1-p} \right\}.$$

In all examples above, when $p = 2$, we may rewrite $\mathcal{F}$ in the form

$$\mathcal{F}(u) = \int_\Omega a(x)|Du|^2\,dx + \int\int |u(x) - u(y)|^2\,j(dx, dy)$$

where the measure $j$ on $\Omega \times \Omega$ is a suitable symmetric combination of Dirac masses laying outside the diagonal $\{x = y\}$. This is the standard integral representation of $\mathcal{F}$ in Dirichlet form (see for instance Mosco [13],[14], for further examples).

**Example 4.2.** Let $x_1, x_2 \in \mathbf{R}$, let $S_1, S_2$ be the 2-dimensional disks in $\mathbf{R}^3$

$$S_1 = \big\{(x, y, z) \;:\; x = x_1,\; y^2 + z^2 < 1\big\}$$
$$S_2 = \big\{(x, y, z) \;:\; x = x_2,\; y^2 + z^2 < 1\big\},$$

let $\Gamma_0, \Gamma_1, \Gamma_2$ be the 1-dimensional segments

$$\Gamma_0 = \big\{(x, y, z) \;:\; x \in [x_1, x_2],\; y = z = 0\big\}$$
$$\Gamma_1 = \big\{(x, y, z) \;:\; x = x_1,\; |y| < 1/2,\; z = 0\big\}$$
$$\Gamma_2 = \big\{(x, y, z) \;:\; x = x_2,\; |y| < 1/2,\; z = 0\big\},$$

let $\Omega$ be an open set containing $S_1 \cup S_2 \cup \Gamma_0$ (see Figure 4.3), and let

$$\mu = H^3 \llcorner \Omega + H^2 \llcorner (S_1 \cup S_2) + H^1 \llcorner (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2).$$

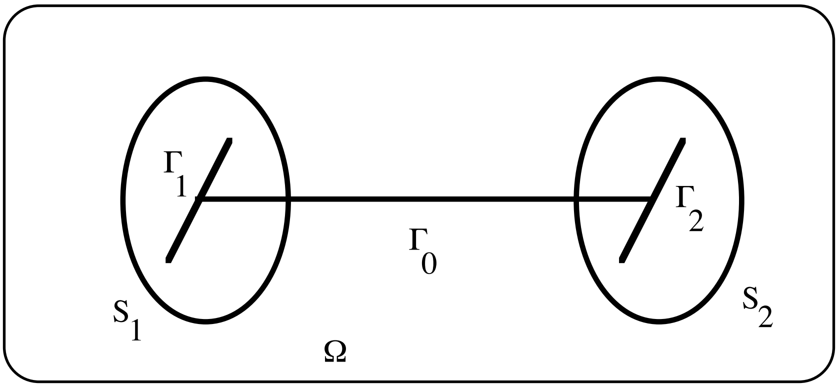


Figure 4.3.

Then we have, with $f(x, z) = |z|^p$,

$$\overline{F}(v) = \int_\Omega |Dv|^p\,dx + \int_{S_1\cup S_2} |D_\tau v|^p\,dH^2 + \int_{\Gamma_0\cup\Gamma_1\cup\Gamma_2} |D_\tau v|^p\,dH^1.$$

In order to characterize the functional $\mathcal{F}$ through formula (3.5), we remark that every $u \in W^{1,p}(\Omega)$ with $\mathcal{F}(u) < +\infty$ belongs to $W^{1,p}(S_1 \cup S_2)$, hence to $W^{1,p}(\Gamma_1 \cup \Gamma_2)$. Hence $u$ is defined at the points $\xi_1 = (x_1, 0, 0)$ and $\xi_2 = (x_2, 0, 0)$, and the equality $u = v$ in formula (3.5) occurs on $S_1 \cup S_2$, on $\Gamma_1 \cup \Gamma_2$, and at $\xi_1$ and $\xi_2$. We obtain

$$\mathcal{F}(u) = \int_\Omega |Du|^p\,dx + \int_{S_1\cup S_2} |D_\tau u|^p\,dH^2 + \int_{\Gamma_1\cup\Gamma_2} |D_\tau u|^p\,dH^1 + \frac{|u(\xi_2) - u(\xi_1)|^p}{|x_2 - x_1|^{p-1}}.$$

We notice that the elimination of the two disks $S_1$ and $S_2$ from the structure would make the set $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ of capacity zero, so that $\mathcal{F}$ would become

$$\mathcal{F}(u) = \int_\Omega |Du|^p \, dx.$$

Analogously, the elimination of the two junctions $\Gamma_1$ and $\Gamma_2$ would make $u$ and $v$ completely independent on $\Gamma_0$, so that

$$\mathcal{F}(u) = \int_\Omega |Du|^p \, dx + \int_{S_1 \cup S_2} |D_\tau u|^p \, dH^2.$$

**Example 4.3.** Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ ($n \geq 2$), let $A \subset\subset \Omega$ be a smooth open set, and let $S$ be a smooth $(n-1)$-dimensional manifold intersecting $\partial A$ transversally (see Figure 4.4).

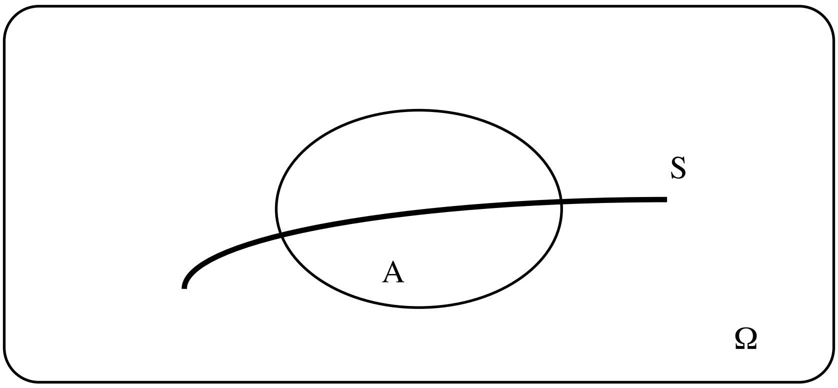


Figure 4.4.

Take $f(x,z) = |z|^p$ and

$$\mu = H^n \llcorner (\Omega \setminus A) + H^{n-1} \llcorner S.$$

Then, by Theorem 3.1 we have for every $v \in W^{1,p}_\mu$

$$\overline{F}(v) = \int_{\Omega \setminus A} |Dv|^p \, dx + \int_S |D_\tau u|^p \, dH^{n-1}.$$

Moreover, if $u \in W^{1,p}(\Omega \setminus A)$ is such that $\mathcal{F}(u) < +\infty$, then $u$ admits a trace on $S \cap (\Omega \setminus A)$ which is in $W^{1,p}(\Omega \setminus A)$, so that equality $u = v$ in formula (3.5) occurs on $S \cap (\Omega \setminus A)$, hence on $S \cap \partial A$, whereas $u$ and $v$ are completely independent on $S \cap A$. Therefore we obtain

$$\mathcal{F}(u) = \int_{\Omega \setminus A} |Du|^p \, dx + \int_{S \cap (\Omega \setminus A)} |D_\tau u|^p \, dH^{n-1} + \inf \left\{ \int_{S \cap A} |D_\tau v|^p \, dH^{n-1} \ : \ v = u \text{ on } S \cap \partial A \right\}.$$

Let us take, for instance, $n = 3$ and $p = 2$. Then the surface $S$ is divided into two parts $S_1 = S \cap A$ and $S_2 = S \setminus A$, separated by the curve $\Gamma = S \cap \partial A$. As an element of $H^1(S_2)$, the function $u$ has a trace on $\Gamma$, still denoted by $u$. Then the solution $v \in H^1(S_1)$ of the problem

$$\min \left\{ \int_{S_1} |D_\tau v|^2 \, dH^2 \ : \ v = u \text{ on } \Gamma \right\}$$

can be represented in terms of a function $G : \Gamma \times \Gamma \to \mathbf{R}$ (Poisson kernel), i.e.,

$$v(x) = \int_\Gamma G(x,y) u(y) \, dH^1(y).$$

Therefore, denoting by $\gamma$ the vector tangential to $S_1$ but normal to $\Gamma$ (pointing outward), and setting

$$K(x,y) = -\frac{1}{2} \frac{\partial G}{\partial \gamma}$$

we get

$$\begin{aligned}\int_{S_1} |D_\tau v|^2 \, dH^2 &= \int_\Gamma v \frac{\partial v}{\partial \gamma} \, dH^1 \\ &= -2 \int_{\Gamma\times\Gamma} K(x,y) u(x) u(y) \, dH^1(x) \, dH^1(y) \\ &= \int_{\Gamma\times\Gamma} K(x,y) |u(x) - u(y)|^2 \, dH^1(x) \, dH^1(y).\end{aligned}$$

Hence we may write $\mathcal{F}$ as a traditional Dirichlet form

$$\mathcal{F}(u) = \int_{\Omega\setminus A} |Du|^2 \, dx + \int_{S\setminus A} |D_\tau u|^2 \, dH^{n-1} + \int_{\Gamma\times\Gamma} K(x,y) |u(x) - u(y)|^2 \, dH^1(x) \, dH^1(y).$$

**Example 4.4.** Consider an elastic 2-dimensional medium on $S$, where $S$ is a smooth 2-dimensional manifold of $\mathbf{R}^3$. Let $\mu$ be the measure $H^2 \llcorner S$, and as energy density we take an integrand

$$f(x,z) = g(x, z^\diamond)$$

where $z^\diamond$ denotes the symmetric part of the $3\times 3$ matrix $z$. We assume that $g(x,\cdot)$ is convex and

$$c_1 |\xi|^p \le g(x,\xi) \le c_2 \big(1 + |\xi|^p\big)$$

for every $x \in \mathbf{R}^3$ and every symmetric $3\times 3$ matrix $\xi$. It is easy to check that the results of previous sections still hold if the coerciveness assumption (3.3) is weakened as follows:

$$c_1 \int |Du|^p \, d\mu \le \int f(x, Du) \, d\mu \le c_2 \int \big(1 + |Du|^p\big) \, d\mu$$

for every $u \in C^1(\mathbf{R}^n; \mathbf{R}^m)$. By Theorem 3.1 we have that

$$\overline{F}(u) = \int_S f_S(x, D_S u) \, dH^2,$$

where $D_S$ is the tangential gradient on $S$ and $f_S$ is given by

$$\begin{aligned}f_S(x,z) &= \inf\big\{ f(x, z + a\otimes\nu) \ : \ a \in \mathbf{R}^3 \big\} \\ &= \inf\big\{ g\big(x, z^\diamond + (a\otimes\nu)^\diamond\big) \ : \ a \in \mathbf{R}^3 \big\}\end{aligned}$$

where $\nu$ denotes the normal to $S$ at $x$. As

$$z^\diamond = (I - \nu\otimes\nu) z^\diamond (I - \nu\otimes\nu) + \big[(2z^\diamond\nu - (\nu z^\diamond \nu)\nu)\otimes\nu\big]^\diamond,$$

it is easy to see that $f_S(x,\cdot)$ actually depends only on the matrix

$$z_S^\diamond = (I - \nu\otimes\nu) z^\diamond (I - \nu\otimes\nu).$$

For instance, in the linear isotropic model, where

$$g(x,\xi) = \frac{\lambda}{2} |\operatorname{Tr} \xi|^2 + \mu |\xi|^2, \tag{4.1}$$

we obtain the usual 2-dimensional membrane energy density (see Landau and Lifchitz [10])

$$f_S(x,z) = \frac{\lambda\mu}{\lambda + 2\mu} |\operatorname{Tr} z_S^\diamond|^2 + \mu |z_S^\diamond|^2.$$

Analogously, for an elastic string modellized by a smooth 1-dimensional manifold $S$ we obtain

$$\overline{F}(u) = \int_S f_S(x, D_S u) \, dH^1$$

where, denoting by $\tau$ the tangent vector to $S$,

$$\begin{aligned}f_S(x,z) &= \inf\big\{ f(x, z + \xi) \ : \ \xi\tau = 0 \big\} \\ &= \inf\big\{ g\big(x, z^\diamond + \xi^\diamond\big) \ : \ \xi\tau = 0 \big\}.\end{aligned}$$

For instance, in the linear isotropic case (4.1) we get

$$f_S(x,z) = \frac{\mu(3\lambda + 2\mu)}{2(\lambda + \mu)} |z\tau\cdot\tau|^2.$$

# 5. Appendix

## 5.1. A Convex Analysis Lemma

In this section we prove under very general assumptions a convex analysis result which we found in the literature under more restrictive hypotheses (see for instance Castaing and Valadier [6]).

**Theorem 5.1.** *Let $X, Y$ be Banach spaces, let $A : X \to Y$ be a linear operator with dense domain $D(A)$, and let $f : Y \to ]-\infty, +\infty]$ be a convex function which we assume to be continuous in at least a point of the image of $A$. Then we have for every $x^* \in X^*$*

$$(f \circ A)^*(x^*) = \inf\{f^*(y^*) \ : \ A^*y^* = x^*\} \tag{5.1}$$

*and, when the quantities above are finite, the infimum at the right-hand side is achieved.*

**Proof.** As usual, we use the conventions that $(f \circ A)(x) = +\infty$ whenever $x \notin D(A)$, and that $\inf \emptyset = +\infty$. Moreover, it is not restrictive to assume that the point of $A(X)$ where $f$ is continuous is the origin. Let us fix $x^* \in X^*$ and for every $y \in Y$ define

$$h(y) = \inf \big\{f(Ax + y) - \langle x, x^*\rangle\big\}. \tag{5.2}$$

Then we have:

$(i)$ $(f \circ A)^*(x^*) = -h(0)$;
$(ii)$ $h(y) \le f(y)$ for every $y \in Y$ $\big($take $x = 0$ in (5.2)$\big)$;
$(iii)$ $h$ is convex on $Y$ $\big($indeed the function $(x, y) \mapsto f(Ax + y) - \langle x, x^*\rangle$ is convex on $D(A) \times Y\big)$;
$(iv)$
$$\begin{aligned} h^*(y^*) &= \sup_y \sup_x \big[\langle y, y^*\rangle + \langle x, x^*\rangle - f(Ax + y)\big] \\ &= \sup_x \Big[\langle x, x^*\rangle + \sup_y \big(\langle y, y^*\rangle - f(Ax + y)\big)\Big] \\ &= \sup_x \big[\langle x, x^*\rangle - \langle Ax, y^*\rangle + f^*(y^*)\big]. \end{aligned}$$

Let us now prove inequality $\le$ in (5.1). We may assume that the right-hand side is finite, so that there exists $y_0^* \in Y^*$ with $A^*y_0^* = x^*$. This yields, by (iv),

$$h^*(y^*) = f^*(y^*) + \sup_x \langle Ax, y_0^* - y^*\rangle = \begin{cases} f^*(y^*) & \text{if } A^*y^* = x^* \\ +\infty & \text{otherwise.} \end{cases}$$

Hence

$$\inf\{f^*(y^*) \ : \ A^*y^* = x^*\} = \inf h^* = -h^{**}(0) \ge -h(0) = (f \circ A)^*(x^*)$$

which proves inequality $\le$ in (5.1).
Let us now prove the opposite inequality. We may assume that the left-hand side is finite, so that by i), $h(0) > -\infty$. As $h$ is convex, $h \le f$, and $f$ is continuous at 0, we have that $h$ is continuous and subdifferentiable at 0. In particular, there exists $z^*$ such that

$$h^*(z^*) = -h(0) < +\infty,$$

and so, by (iv),

$$f^*(z^*) < +\infty, \qquad \sup_x \big[\langle x, x^*\rangle - \langle Ax, z^*\rangle\big] < +\infty.$$

Hence

$$\sup_{\|x\| \le 1} \big|\langle Ax, z^*\rangle\big| \le M + \|x^*\| < +\infty,$$

which implies that $z^* \in D(A^*)$. Therefore, we can write $\langle Ax, z^*\rangle = \langle x, A^*z^*\rangle$, so that (since $D(A)$ is dense in $X$)

$$h^*(z^*) = f^*(z^*) + \sup_x \langle x, x^* - A^*z^*\rangle = \begin{cases} f^*(z^*) & \text{if } A^*z^* = x^* \\ +\infty & \text{otherwise.} \end{cases}$$

Since $h^*(z^*) < +\infty$, we have $A^*z^* = x^*$, and so

$$(f \circ A)^*(x^*) = -h(0) = f^*(z^*) \geq \inf\{f^*(y^*) \ : \ A^*y^* = x^*\}. \quad \blacksquare$$

### 5.2. Different Notions of Tangent Space

In the literature we found different notions of tangent space relative to a measure (see De Giorgi [9], Simon [16]). In [16] the following definition was proposed, using blow-up of the measure $\mu$. Let $x_0 \in \mathbf{R}^n$ and for every $\rho > 0$ define the measure $\mu_\rho^{x_0}$ by

$$\langle \mu_\rho^{x_0}, \varphi\rangle = \frac{1}{\mu(B_\rho(x_0))} \int \varphi\left(\frac{x - x_0}{\rho}\right) d\mu \qquad (\varphi \in C_c^0(\mathbf{R}^n)). \tag{5.3}$$

Then the $k$-dimensional subspace $P$ of $\mathbf{R}^n$ is said to be tangent to $\mu$ at $x_0$ if there exist $\rho_h \downarrow 0$ and $\theta > 0$ such that

$$\lim_{h\to+\infty} \langle \mu_{\rho_h}^{x_0}, \varphi\rangle = \theta \int_P \varphi \, dH^k \qquad \text{for every } \varphi \in C_c^0(\mathbf{R}^n). \tag{5.4}$$

Let us notice that the existence of such a $P$, unlike our notion $T_\mu^p$, is not guarranted (in case $\mu = H^k \llcorner S$, the existence $\mu$-a.e. of $P$ is equivalent to the $k$-rectifiability of the set $S$, see [16]). The link between the two notions is given through the following result.

**Lemma 5.2.** *Assume that there exists a $\mu$-measurable multifunction $P(x)$ such that (5.4) holds $\mu$-a.e. Then for every $p \in [1, +\infty[$, we have*

$$T_\mu^p(x) \subset P(x) \qquad \text{for } \mu\text{-a.e. } x \in \mathbf{R}^n.$$

**Proof.** Let $\Phi \in X_\mu^{p'}$. We only need to show that $\Phi(x_0) \in T_\mu^p(x_0)$ $\mu$-a.e.. By definition of the space $X_\mu^{p'}$ we know that $m = |\operatorname{div}(\Phi\mu)|$ is a $\mu$-absolutely continuous Radon measure. Hence, for $\mu$-a.e. $x_0$ we have

$$\limsup_{\rho\to 0} \frac{m(B_\rho(x_0))}{\mu(B_\rho(x_0))} < \infty. \tag{5.5}$$

Let $\Psi$ be a smooth function such that $\operatorname{spt}\Psi \subset B_1(0)$ and set $\psi_\rho(x) = \Psi\left(\frac{x-x_0}{\rho}\right)$, $M = \sup|\Psi|$. We have

$$\begin{aligned} m((B_\rho(x_0)) \geq & \frac{1}{M} \left| \int \Phi \cdot D\psi_\rho \, d\mu \right| \\ \geq & \frac{1}{\rho M} \left| \int \Phi \cdot D\psi_\rho\left(\frac{x - x_0}{\rho}\right) d\mu \right|. \end{aligned}$$

By (5.5) and (5.4), we deduce that

$$\begin{aligned} 0 = & \lim_k \frac{1}{\mu(B_{\rho_k}(x_0))} \int \Phi \cdot D\psi_{\rho_k}\left(\frac{x - x_0}{\rho_k}\right) d\mu \\ = & \Phi(x_0) \cdot \theta(x_0) \int_{P(x_0)} D\Psi(y) \, H^{k(x_0)}(dy). \end{aligned} \tag{5.6}$$

Now integrating by parts the orthogonal projection on $P(x_0)$ of $D\Psi$, we find easily that $(P(x_0))^\perp$ is spanned by the set

$$\Big\{\int_{P(x_0)} D\Psi(y)\, H^{k(x_0)}(dy)\ :\ \Psi\in\mathcal{D}(\mathbf{R}^n),\ \mathrm{spt}\,\Psi\subset B_1(x_0)\Big\}.$$

Hence by (5.6), we have $\Phi(x_0)\in P(x_0)$ for $\mu$-a.e. $x_0$. ■

**Remark 5.3.** The inclusion $T^p_\mu(x)\subset P(x)$ can be strict; indeed, taking $\alpha=1$ in Example 2.5, we get $P(x)=\mathbf{R}$ on all points of $C$ with density 1 (with respect to the Lebesgue measure), whereas $T^p_\mu(x)=0$ for all $x\in\mathbf{R}$.

Now we justify the results stated in Example 2.4.

**Corollary 5.4.** *Let $S$ be a $C^2$ manifold in $\mathbf{R}^n$ of dimension $k\le n$, let $T_S(x)$ be the tangent space at every $x\in S$, and let $\mu=H^k\llcorner S$. Then for every $p\in[1,+\infty[$ we have*

$$T^p_\mu(x)=T_S(x)\qquad \mu\text{-a.e.}$$

**Proof.** By Lemma 5.2 and the rectifiability of $S$, we have $T^p_\mu(x)\subset T_S(x)$ for $\mu$-a.e. $x$. Using integration by parts on $S$, we find that every $C^2$ vector field $\Psi$ such that $\Psi(x)\in T_S(x)$ for every $x\in S$ and $\Psi=0$ on $\partial S$ belongs to $X^{p'}_\mu$ (for every $p\ge 1$). Hence (as $H^k(\partial S)=0$), we have $T_S(x)\subset T^p_\mu(x)$ $\mu$-a.e.. ■

**Corollary 5.5.** *Let $S$ be a finite union of $C^2$ manifolds $S_i$ $(i=1,\dots,N)$; assume that $S_i$ has dimension $k_i$ and that the measures $\mu_i=H^{k_i}\llcorner S_i$ are mutually singular. Setting $\mu=\sum_i\mu_i$ and denoting by $T_{S_i}(x)$ the tangent space to $S_i$ at $x$, we have for every $p\in[1,+\infty[$*

$$T^p_\mu(x)=T_{S_i}(x)\quad \mu_i\text{-a.e.}.$$

**Proof.** Since $\mu\ge\mu_i$, it is easy to see that $T^p_{\mu_i}(x)\subset T^p_\mu(x)$ $\mu_i$-a.e while, by Corollary 5.4, $T^p_{\mu_i}(x)=T_{S_i}(x)$ $\mu_i$-a.e.. To prove the opposite inclusion, we consider the blown-up measure $\mu^x_\rho$ defined in (5.3). We have

$$\mu^x_\rho=\sum_j\frac{\mu_j(B_\rho(x))}{\mu(B_\rho(x))}(\mu_j)^x_\rho.$$

Since the measures $\mu_j$ are mutually singular, we have that for every $i$ and for $\mu_i$-a.e. $x\in\mathbf{R}^n$

$$\lim_{\rho\to 0}\frac{\mu_j(B_\rho(x))}{\mu(B_\rho(x))}=\begin{cases}1 & \text{if } j=i\\ 0 & \text{if } i\ne j.\end{cases}$$

Hence, for $\mu_i$-a.e. $x$, the weak limit of $\mu^x_\rho$ in the sense of measures is the same as that of $(\mu_i)^x_\rho$, which, by Corollary 5.4 is given by (5.4) with $P=T_{S_i}(x)$ and $k=k_i$. We deduce that $T_{S_i(x)}$ is the tangent plane to $\mu$ in the sense of the definition (5.4), and so, by Lemma 5.2, we conclude that $S_i(x)$ contains $T_\mu(x)$ for $\mu_i$-a.e. $x$. ■

**Acknowledgements.** The first author wishes to thank the Department of Mathematics of Università di Pisa where this paper was written; his research is part of the EEC-SCIENCE project "EURHomogenization", contract SC1-CT91-0732. The second author warmly acknowledges the hospitality of the Department of Mathematics of the Université de Toulon, where this paper was initiated; his research is part of the EEC-SCIENCE project "EURHomogenization", contract SC1-CT91-0732, and of the EEC-HCM project "Phase Transition Problems and Singular Perturbations", contract CHRX-CT94-0608.

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