REGULAR APPROXIMATION OF FREE-DISCONTINUITY PROBLEMS

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We consider a class of smooth local non convex functionals defined on $W^{2,2}(\Omega)$, depending on a small parameter ε, and we prove that they converge, as ε tends to 0, to a functional $F(u,\Omega)$ with a bulk density depending on the gradient of u and a surface energy concentrated on the jump set of u. This provides a new alternative to the approximation of free discontinuity problems, which applies in particular to the Mumford-Shah model.

1. Introduction

Many problems in computer vision,[17,19] in fracture mechanics or in mathematical physics[15] are modelized by variational problems in which the unknown function admits discontinuities. Part of the energy is a volume density, the other part being concentrated along the (free) discontinuity zone. Hence, Ω being an open connected set of $\mathbb{R}^n$, one has to minimize :

$$\inf\left\{\int_{\Omega\setminus\mathcal{S}_u} f(u,\nabla u)\,dx+\int_{\mathcal{S}_u}\varphi(u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}\ ,\ u\in SBV(\Omega)\right\}$$

where u^-, u^+ are the upper and lower approximative limits of u and ν_u is the unit normal to $\mathcal{S}_u$ (for the definition of the space $SBV(\Omega)$ and related topics we refer for instance to Ambrosio[4])

In particular in image segmentation problems one has to minimize, for a given data $g\in L^\infty(\Omega)$:

$$\inf_u\left\{MS(u,\Omega)+\int_\Omega |u-g|^2\,dx,\ u\in SBV(\Omega)\right\},$$

where the principal part of the energy is the Mumford and Shah functional defined by:

$$MS(u,\Omega)=\int_{\Omega\setminus\mathcal{S}_u}|\nabla u|^2\,dx+\mathcal{H}^{n-1}(\mathcal{S}_u\cap\Omega).$$

Since the first existence result obtained by De Giorgi, Carriero and Leaci,[12] a lot of work has been directed in view of numerical approaches or in view of defining a parabolic evolution model. These questions are highly difficult because of the lack of convexity and regularity mainly due to the term $\mathcal{H}^{n-1}(\mathcal{S}_u)$. In fact we need to define a smooth approximation of the Mumford and Shah functional or, more generally, of functionals of the type

$$a\int_{\Omega\setminus\mathcal{S}_u}|\nabla u|^2\,dx+b\int_{\mathcal{S}_u}|u^+-u^-|^\gamma\,d\mathcal{H}^{n-1}. \tag{1.1}$$

The Mumford and Shah functional corresponds to the case $\gamma=0$. In this case, an approximation has been proposed by Ambrosio and Tortorelli[3]:

$$\int_\Omega v\,|\nabla u|^2\,dx+\frac{1}{2}\int_\Omega\left(\varepsilon|\nabla v|^2+\frac{1}{\varepsilon}(1-v)^2\right)dx,$$

where the auxiliary variable v plays as a control on the jump set $\mathcal{S}_u$. As shown by Bellettini and Coscia,[6] the functional involved here can be discretized using a finite elements method where the discretization step has been suitably adjusted according to ε. Another approach connected with finite differences discretization (see Chambolle[11]) leads to consider non-local approximations. In particular the functional

$$\frac{1}{\varepsilon}\int_{\Omega\times\Omega}\arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right)\mathrm{e}^{-|\xi|^2}\,d\xi\,dx$$

has been suggested by De Giorgi (see Gobbino[16] for a mathematical justification). In the same spirit, the functional

$$\frac{1}{\varepsilon}\int_\Omega\arctan\left(\varepsilon\fint_{B(x,\varepsilon)\cap\Omega}|\nabla u|^2\,dy\right)dx$$

has been proposed by Braides and Dal Maso.[10]

Recently, Alicandro, Braides and Gelli[1,2] have suggested an approach very similar to ours. It consists in considering an elliptic regularization with higher order terms. In fact, their result appears to be a particular case of our main theorem (the case when $\gamma=1/2$). They also deal with the case $\gamma=0$ but using a discontinuous potential.

For a survey on the approximation of free-discontinuity problems, we refer to Braides.[9]

In this paper we propose a family of functionals F_ε which are regular approximations of every functional of type (1.1), for every exponent γ ranging over $[0,1)$. The spirit of the method originates from a previous work concerning capillary equilibrium of droplets, [8] where the main idea was to consider degenerated potentials at infinity.

More precisely, we define the family (F_ε), indexed by the positive real parameter ε, by setting ($p\vee q$ denotes $\max\{p,q\}$)

$$F_\varepsilon(u,\Omega):=\begin{cases}\displaystyle\int_\Omega\frac{|\nabla u|^2}{1+(\varepsilon\,|\nabla u|)^p}\,dx+\varepsilon^{\frac{3p}{p-1}\vee 4}\int_\Omega\|\nabla^2 u\|^2\,dx & \text{if } u\in W^{2,2}(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{1.2}$$

Here p is a real parameter ($p > 1$ and $p \neq 4$) which rules the behaviour of the potential $W(t) := \dfrac{t^2}{1+t^p}$ at infinity. Accordingly we reach every functional of type (1.1) for $\gamma \in [0,1)$. Precisely, in the one-dimensional case, we prove that F_ε converges in Mosco-sense, as ε tends to zero, to the functional F defined on L^2 by:

$$F(u,\Omega) := \begin{cases} \displaystyle\int_{\Omega\setminus\mathcal{S}_u} |\nabla u|^2\,dx + k_p \int_{\mathcal{S}_u} |u^+ - u^-|^{\gamma_p}\,d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

where the real parameters γ_p and k_p depend explicitely on p (Γ denotes the Euler function):

$$\gamma_p := \frac{(4-p)_+}{2+p} \tag{1.4}$$

$$k_p := \begin{cases} \left(\dfrac{2+p}{4-p}\right)\left(\dfrac{2\sqrt{\pi}\,\Gamma\left(\frac{2+p}{2p-2}\right)}{(p-1)\,\Gamma\left(\frac{2p+1}{2p-2}\right)}\right)^{\frac{2p-2}{2+p}} & \text{if } 1 < p < 4, \\ 4\displaystyle\int_0^\infty \sqrt{W(s)}\,ds = \frac{1}{p\sqrt{\pi}}\,\Gamma\left(\tfrac{2}{p}\right)\,\Gamma\left(\tfrac{4-p}{2p}\right) & \text{if } p > 4. \end{cases} \tag{1.5}$$

Moreover we prove that, given a sequence (u_ε) which is bounded in $L^2(\Omega)$ and has uniformly bounded energy $F_\varepsilon(u_\varepsilon,\Omega)$, then it is precompact in $L^q(\Omega)$ for all $q < 2$ (this is optimal in the case $p > 4$) and for all $q < +\infty$ in the case $p < 4$.

The main limitation of this paper is that we encompass only the one-dimensional case. However it seems that, when the potential is increasing, i.e. when $p \leq 2$, the convergence of F_ε to F can be deduced in higher dimension by using the so called slicing method (see for instance [7]). The case $p > 2$ is more intricate and remains completely open although we think that the convergence result still holds true.

In the next section, we recall some general notations and state our main result (Theorem 2.1), and its application to image segmentation problems (Corollary 2.2).

In section 3, we present some preliminary results about the convergence of Dirichlet-Neumann problems related to F_ε. The last two sections are devoted to the proof of the main theorem. The compactness result (*i*) is proved in section 4; the lower-bound inequality (*ii*) and the upper-bound inequality (*iii*) are proved in section 5.

2. Notations and main results

2.1. *Notations*

Although all results presented in this paper are proved only when Ω is a bounded interval of $\mathbb{R}$, we will use the notations related to a general open subset Ω of $\mathbb{R}^n$

$(n \geq 1)$. For all element u of the classical Sobolev space $W^{2,2}(\Omega)$, we denote by ∇u the a.e defined gradient and by $\|\nabla^2 u\|$ the euclidean norm of the Hessian matrix.

The space $BV(\Omega)$ is the space of all functions $u \in L^1(\Omega)$ whose first order distribution derivatives are bounded Radon measures on Ω. For such functions, the Radon Nykodim derivative of the vector measure Du with respect to the Lebesgue measure, denoted $\nabla u(x)$ (or simply u' if $n = 1$), coincides for $\mathcal{L}^n$-a.e.x with the approximate gradient of u. Moreover the approximate upper and lower limits, denoted $u^+(x)$ and $u^-(x)$, agree as Borel functions on all $\mathbb{R}^n$ except a subset of vanishing n-dimensional Lebesgue measure. The jump set $\mathcal{S}_u := \{x \mid u^-(x) \neq u^+(x)\}$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable. In the case $n = 1$, this means that $\mathcal{S}_u$ is at most countable and we can choose a reprentative of u which is bounded, continuous on $\Omega \setminus \mathcal{S}_u$ and admits at every $x \in \mathcal{S}_u$ left and right-hand limits $u(x \pm 0)$ (hence $u^+(x) = \max\{u(x+0), u(x-0)\}$). For further details on the general theory of BV functions, we refer for example to the book by Gariepy-Evans,[14] chap. 5.

The space $SBV(\Omega)$ is the space of special functions with bounded variation introduced by De Giorgi and Ambrosio[13]: it consists of all functions u in $BV(\Omega)$ such that the singular part of Du is carried by $\mathcal{S}_u$. In the one dimensional case, this is equivalent to say that the measure Du has the form $Du = u'(x)dx + \sum_{x \in \mathcal{S}_u} (u(x+0) - u(x-0))\, \delta_x$, where δ_x denotes the Dirac mass at x.

Some specific notations, used all along this paper, have already been defined in section 1 (the definitions of the energies F_ε, F and related the parameters p, γ_p, k_p are given in (1.2) to (1.5)).

2.2. *Main results*

Theorem 2.1 *Let Ω be an open bounded interval in $\mathbb{R}$. We have the following compactness and convergence results :*

(i) Let (u_ε) be a bounded sequence in $L^2(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon, \Omega) < +\infty$. We have :

- *if $p > 4$, (u_ε) is strongly relatively compact in $L^q(\Omega)$ for all $q < 2$,*
- *if $p < 4$, (u_ε) is strongly relatively compact in $L^q(\Omega)$ for all $q < +\infty$,*

(ii) for every sequence (u_ε) weakly converging to u in $L^2(\Omega)$, we have

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \geq F(u, \Omega) ,$$

(iii) for every function $u \in SBV(\Omega)$, there exists u_ε strongly converging to u in $L^2(\Omega)$ such that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \leq F(u, \Omega) .$$

Remarks

a) Following Mosco (see Attouch[5] for example), the assertions ii) and iii) of Theorem 2.1 are equivalent to the Mosco-convergence of F_ε in $L^2(\Omega)$.

b) Assertion i) can be made precise as follows: in the case $p < 4$, the sequence (u_ε) is bounded in $W^{1,1}(\Omega)$ (see Lemma 4.5)). This estimate fails in the case $p > 4$ as well as strong precompactness in $L^2(\Omega)$ (indeed the sequence $u_\varepsilon(x) := \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon^2})$ where φ is smooth and compactly supported, is bounded in energy whereas $|u_\varepsilon|^2$ converges weakly star to a Dirac mass). However, in this case, we know that any cluster point u is continuous except possibly on a finite set D and that uniform convergence will hold on every compact subset of $\Omega \setminus D$ (see Lemma 4.4).

c) It appears in the proof of iii) (see subsection 5.2) that the approximating sequence (u_ε) can be choosen so that $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$.

An immediate application of Theorem 2.1 is that, for every $g \in L^2(\Omega)$, the free discontinuity problem:

$$(\mathcal{P}) \qquad \inf_u \left\{ F(u,\Omega) + \int_\Omega |u-g|^2\,dx \right\}$$

can be obtained as a limit of

$$(\mathcal{P}_\varepsilon) \qquad \inf_u \left\{ F_\varepsilon(u,\Omega) + \int_\Omega |u-g|^2\,dx \right\}.$$

In particular, for $p > 4$, we obtain, using $(\mathcal{P}_\varepsilon)$, a regular approximation of the image segmentation problem $(\mathcal{P})$.

Corollary 2.2 *Let $p > 1$, $p \neq 4$, and $g \in L^2(\Omega)$. Then*

$$\lim_{\varepsilon\to 0} \Big(\inf(\mathcal{P}_\varepsilon) \Big) = \inf(\mathcal{P}).$$

Moreover, every sequence (u_ε) such that

$$F_\varepsilon(u_\varepsilon,\Omega) \ + \ \int_\Omega |u_\varepsilon - g|^2\,dx \ - \ \inf(\mathcal{P}_\varepsilon) \ \longrightarrow \ 0$$

is strongly relatively compact in $L^q(\Omega)$ for all $q \in [1,2)$ (for all $q \in [1,\infty)$ if $p < 4$), and any cluster point u is a solution of $(\mathcal{P})$.

Remark Thanks to the compactness assertion of Theorem 2.1 (and part c) in the subsequent remarks), when $g \in L^\infty(\Omega)$ it is possible to restate Corollary 2.2 with any variant of $(\mathcal{P})$ obtained by adding to the energy the perturbation term $\int_\Omega V(u)dx$ where $V : \Omega \to [0,+\infty)$ is any continuous (possibly nonconvex) potential. Such a variant could be useful for example to select solutions taking values in a prescribed set (grey levels).

Proof. Let (u_ε) be a minimizing sequence for $(\mathcal{P}_\varepsilon)$. Clearly, (u_ε) is bounded in $L^2(\Omega)$ and by assertion i) of Theorem 2.1, (u_ε) is strongly relatively compact in $L^q(\Omega)$ for all $q \in [1,2)$ (for all $q \in [1,\infty)$ if $p < 4$). We denote by u a cluster point of u_ε. Then assertion ii) of Theorem 2.1 yields:

$$\begin{aligned} \liminf_{\varepsilon\to 0} \Big(\inf(\mathcal{P}_\varepsilon) \Big) \ &\geq \ \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega) + \liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon - g|^2 dx \\ &\geq \ F(u,\Omega) + \int_\Omega |u-g|^2\,dx \geq \inf(\mathcal{P}). \end{aligned} \tag{2.1}$$

Conversely, let v be admissible for $(\mathcal{P})$. By Theorem 2.1 iii), there exists a sequence (v_ε) such that $v_\varepsilon \to v$ in $L^2(\Omega)$ and $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, \Omega) \le F(v,\Omega)$. Therefore

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Big(\inf(\mathcal{P}_\varepsilon)\Big) &\le \limsup_{\varepsilon\to 0}\Big(F_\varepsilon(v_\varepsilon,\Omega)+\int_\Omega |v_\varepsilon - g|^2\,dx\Big)\\ &\le F(v,\Omega)+\int_\Omega |v-g|^2\,dx \le \inf(\mathcal{P}). \qquad (2.2)\end{aligned}$$

Then by (2.1) and (2.2), we have

$$\inf(\mathcal{P}) = \lim_{\varepsilon\to 0}\Big(\inf(\mathcal{P}_\varepsilon)\Big) = F(u,\Omega)+\int_\Omega |u-g|^2\,dx$$

and so the proof is concluded. □

3. Preliminary results

3.1. *Some auxiliary variational problems*

In all this section, we will deal with the following Dirichlet-Neumann problem:

$$I_\varepsilon(\alpha,h,r,s) := \inf_u \left\{F_\varepsilon\left(u,(-\tfrac{h}{2},\tfrac{h}{2})\right);\ u(-\tfrac{h}{2})=0,\ u(\tfrac{h}{2})=\alpha,\ u'(-\tfrac{h}{2})=r,\ u'(\tfrac{h}{2})=s\right\} \qquad (3.1)$$

where the arguments α, h, r, s are real numbers ($h>0$). It will be convenient to express the functional $I_\varepsilon(\alpha,h,r,s)$ in term of the lower order functional G_ε defined by:

$$G_\varepsilon(v,\Omega) := \begin{cases} \displaystyle\int_\Omega \frac{v^2}{1+(\varepsilon\, v)^p}\,dx + \varepsilon^{\frac{3p}{p-1}\vee 4}\int_\Omega v'^2\,dx & \text{if } v\in H^1(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \qquad (3.2)$$

Lemma 3.1 *For every α, h, r, s and every $\varepsilon>0$, the following equalities hold true*

$$I_\varepsilon(\alpha,h,r,s) = \inf\left\{G_\varepsilon(v,(-\tfrac{h}{2},\tfrac{h}{2}));\ \int_{-\frac{h}{2}}^{\frac{h}{2}} v\,dx=\alpha,\ v(-\tfrac{h}{2})=r,\ v(\tfrac{h}{2})=s\right\}. \qquad (3.3)$$

$$I_\varepsilon(\alpha,h,0,0) = \inf\left\{G_\varepsilon(v,(-\tfrac{h}{2},\tfrac{h}{2}));\ \int_{-\frac{h}{2}}^{\frac{h}{2}} v\,dx=\alpha,\ v(-\tfrac{h}{2})=v(\tfrac{h}{2})=0,\ v \textit{ even and monotone on } (-\tfrac{h}{2},0)\right\}. \qquad (3.4)$$

Moreover the minimum in the right-hand sides of (3.3) or (3.4) is achieved.

Remark It follows from Lemma 3.1 that there exists a monotone solution for the homogeneous Dirichlet-Neumann problem (3.1) where $r=s=0$ (simply take $u_\varepsilon(x) := \int_{-h/2}^{x} w_\varepsilon(s)\,ds$, where w_ε is solution of (3.4))

Proof. The first equality (3.3) is obtained by setting $v = u'$ in the right-hand side of (3.1). Since problem (3.3) is coercive in $H^1(-\frac{h}{2}, \frac{h}{2})$, the existence of a solution v_ε follows classically by applying the direct method of the calculus of variations.

Let us assume now that $r = s = 0$. In this case we may consider the symetric rearrangement of $|v_\varepsilon|$ (see Mossino[18]) on $[-\frac{h}{2}, \frac{h}{2}]$. Let us denote it $\tilde{v}_\varepsilon$. Without loss of generality we assume that $\alpha \geq 0$. Then we may choose $b_\varepsilon \in [0, \frac{h}{2})$ such that $\int_{b_\varepsilon}^{h/2} \tilde{v}_\varepsilon(s)\, ds = \alpha/2$. Setting $w_\varepsilon(x) = \tilde{v}_\varepsilon(|x| + b_\varepsilon)$, we obtain an admissible function for (3.4). Therefore, from the classical properties of the rearrangement, we deduce

$$\begin{aligned} I_\varepsilon(\alpha, h, 0, 0) &\leq G_\varepsilon(w_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2}))\ \{= G_\varepsilon(\tilde{v}_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2}) \setminus (-b_\varepsilon, b_\varepsilon))\} \\ &\leq G_\varepsilon(\tilde{v}_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) \\ &\leq G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) = I_\varepsilon(\alpha, h, 0, 0)\ . \end{aligned}$$

It follows that w_ε is a solution for both (3.3) and (3.4). □

In section 3.2, we study the asymptotic behaviour of I_ε. In particular, we will show that its limit as ε tends to zero can be expressed as:

$$I_p(\alpha, h) := \inf_u \left\{ F(u, (-\tfrac{h}{2}, \tfrac{h}{2}))\ ;\ u(-\tfrac{h}{2} + 0) = 0,\ u(\tfrac{h}{2} - 0) = \alpha \right\}, \tag{3.5}$$

where F (defined by (1.3)) is the expected limit functional associated with F_ε. We notice first that the minimum value $I_p(\alpha, h)$ can be made explicit as stated in:

Lemma 3.2 *Let $p > 1$ and k_p defined by (1.5). Then for all $\alpha, h > 0$ we have:*

$$I_p(\alpha, h) = \begin{cases} \displaystyle\inf_{\beta \in [0,\alpha]} \left\{ \frac{\beta^2}{h} + k_p\, |\alpha - \beta|^{\gamma_p} \right\} & \text{if } 1 < p < 4, \\ \min\left\{ \dfrac{\alpha^2}{h}, k_p \right\} & \text{if } p > 4. \end{cases} \tag{3.6}$$

Moreover, problem (3.5) admits a monotone solution.

Proof. Let u be an admissible function of (3.5) in $SBV(-\frac{h}{2}, \frac{h}{2})$. Its distributional derivative can be written as

$$u' = v\, dx + \sum_{i \in \mathbb{N}} \beta_i \delta_{x_i} \qquad (x_i \in \mathcal{S}_v \cap (-\tfrac{h}{2}, \tfrac{h}{2})).$$

Let $\beta := \displaystyle\int_{-\frac{h}{2}}^{\frac{h}{2}} v\, dx$. The Dirichlet conditions on $\pm\frac{h}{2}$ on u imply $\displaystyle\sum_{i \in \mathbb{N}} \beta_i = \alpha - \beta$. Applying Jensen's inequality and the sub-additivity of the function $t \longmapsto |t|^{\gamma_p}$ ($\gamma_p \in [0, 1)$), we get:

$$\begin{aligned} F(u, (-\tfrac{h}{2}, \tfrac{h}{2})) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} |v|^2\, dx + k_p \sum_{i \in \mathbb{N}} |\beta_i|^{\gamma_p} \\ &\geq \frac{1}{h} \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} v\, dx \right)^2 + k_p |\alpha - \beta|^{\gamma_p} = \frac{\beta^2}{h} + k_p |\alpha - \beta|^{\gamma_p} \end{aligned}$$

(in the case $\gamma_p = 0$, we make the convention $t^0 = 1$ if $t \neq 0$, and $t^0 = 0$ if $t = 0$). Passing to the infimum with respect to u and β, we obtain

$$I_p(\alpha, h) \geq \inf_{\beta \in [0,\alpha]} \left\{ \frac{\beta^2}{h} + k_p \, |\alpha - \beta|^{\gamma_p} \right\}.$$

Now to prove that this lower bound is optimal, we consider the monotone increasing function defined by:

$$u_0(x) := \frac{\beta_0}{h}\, x + \frac{\beta_0}{2} \text{ if } x \in [-\tfrac{h}{2}, 0]\,, \quad u_0(x) := \frac{\beta_0}{h}\, x + \alpha - \frac{\beta_0}{2} \text{ if } x \in [0, \tfrac{h}{2}],$$

where β_0 minimizes the function $t \mapsto \dfrac{t^2}{h} + k_p \, |\alpha - t|^{\gamma_p}$. Then clearly we have $F(u_0, (-\frac{h}{2}, \frac{h}{2})) = I_p(\alpha, h)$ and u_0 is a monotone solution of (3.5). □

3.2. *Asymptotic behaviour of* I_ε

In this subsection, we begin by proving that the limit of $I_\varepsilon(\alpha, h, 0, 0)$ (homogeneous Dirichlet Neumann conditions) exists and coincides with $I_p(\alpha, h)$. Since the cases $p > 4$ (integrability of $\sqrt{W}$) and $p < 4$ differ strongly, this result is stated separately in Propositions 3.7 and 3.9. The extension to the case $(r, s) \neq (0, 0)$ is deduced in both cases through Proposition 3.10. We begin by some preparation lemmas.

Lemma 3.3 *For all $h, \eta > 0$, for all $\alpha, r, s, \delta, t \in \mathbb{R}$ we have:*

$$I_\varepsilon(-\alpha, h, r, s) = I_\varepsilon(\alpha, h, -r, -s), \tag{3.7}$$

$$I_\varepsilon(\alpha + \delta, h + \eta, r, s) \leq I_\varepsilon(\delta, \eta, r, t) + I_\varepsilon(\alpha, h, t, s). \tag{3.8}$$

Proof. it is a direct consequence of the definition (3.1). □

Lemma 3.4 *For all $h > 0$, the function $\alpha \mapsto I_\varepsilon(\alpha, h, 0, 0)$ is increasing on $\mathbb{R}_+$.*

Proof. Let $\alpha \geq 0$ and $\eta > 0$. By Lemma 3.1, we may choose a solution v_ε for (3.4) such that $G_\varepsilon(v_\varepsilon, (-\frac{h}{2}, \frac{h}{2})) = I_\varepsilon(\alpha + \eta, h, 0, 0)$. Let w_ε be defined by

$$w_\varepsilon(x) := v_\varepsilon(x + t_\varepsilon) \;\text{ if } 0 \leq x \leq \tfrac{h}{2} - t_\varepsilon \quad , \quad w_\varepsilon(x) := 0 \text{ if } |x| > \tfrac{h}{2} - t_\varepsilon \;,$$

where t_ε is chosen in $[0, \frac{h}{2}]$ so that $\int_0^{t_\varepsilon} v_\varepsilon(x)\, dx = \eta/2$ (it exists since $\int_0^{h/2} v_\varepsilon(s)\, ds = (\alpha + \eta)/2$). Then, $\int_{-\frac{h}{2}}^{\frac{h}{2}} w_\varepsilon(x)\, dx = \alpha$ and so $I_\varepsilon(\alpha, h, 0, 0) \leq G_\varepsilon(w_\varepsilon, (-\frac{h}{2}, \frac{h}{2}))$. Noticing that $v_\varepsilon(x) > 0$ for $|x| < t_\varepsilon$, we deduce:

$$\begin{aligned} I_\varepsilon(\alpha, h, 0, 0) \;&\leq\; G_\varepsilon(w_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) = G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) - G_\varepsilon(v_\varepsilon, (-t_\varepsilon, t_\varepsilon)) \\ &<\; G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) = I_\varepsilon(\alpha + \eta, h, 0, 0) \end{aligned}$$

and so the proof is completed. □

Lemma 3.5 *For all $\alpha \in \mathbb{R}$, the function $h \mapsto I_\varepsilon(\alpha, h, 0, 0)$ is non increasing on $\mathbb{R}_+$.*

Proof. It is enough to extend by zero a solution of problem (3.4). □

The following lemma will be crucial to obtain that the limit of $I_\varepsilon(\alpha,h,r,s)$ does not depend on Neumann conditions (r,s) (see Proposition 3.10).

Lemma 3.6 *Let (t_ε) be a sequence in $\mathbb{R}$ satisfying $\varepsilon\, t_\varepsilon \to 0$ if $p>4$ or $\varepsilon^{\frac{p}{p-1}} t_\varepsilon \to 0$ if $p\in(1,4)$. Then, there exist sequences (h_ε) and (δ_ε) such that*

$$h_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0,\ \ \delta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0, \qquad I_\varepsilon(\delta_\varepsilon, h_\varepsilon, 0, t_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof. We assume without loss of generality that $t_\varepsilon > 0$.

If $p>4$, we set $h_\varepsilon := \varepsilon^2$, $v_\varepsilon(x) := t_\varepsilon\, h_\varepsilon^{-1}\, x$ and $\delta_\varepsilon := \frac{1}{2} t_\varepsilon\, h_\varepsilon$. The sequences h_ε and δ_ε clearly tend to 0 and a straightforward computation gives

$$I_\varepsilon(\delta_\varepsilon, h_\varepsilon, 0, t_\varepsilon) \le G_\varepsilon(v_\varepsilon, (0,h_\varepsilon)) \le \frac{4}{3}(\varepsilon\, t_\varepsilon)^2. \tag{3.9}$$

If $1<p<4$, we set

$$v_\varepsilon(x) := \varepsilon^{-\frac{2+p}{p-1}}\, x^{\frac{2}{p}}, \quad h_\varepsilon := \left(t_\varepsilon\, \varepsilon^{\frac{2+p}{p-1}}\right)^{\frac{p}{2}}, \quad \delta_\varepsilon := \frac{p}{2+p}\left(t_\varepsilon\, \varepsilon^{\frac{p}{p-1}}\right)^{\frac{2+p}{2}}$$

so that $v_\varepsilon(h_\varepsilon) = t_\varepsilon$ and $\int_0^{h_\varepsilon} v_\varepsilon\, dx = \delta_\varepsilon$. The sequences h_ε and δ_ε clearly tend to 0 and a straightforward computation gives

$$I_\varepsilon(\delta_\varepsilon, h_\varepsilon, 0, t_\varepsilon) \le G_\varepsilon(v_\varepsilon, (0,h_\varepsilon)) \le \frac{p^2+4}{p(4-p)}\left(t_\varepsilon\, \varepsilon^{\frac{p}{p-1}}\right)^{\frac{4-p}{2}}. \tag{3.10}$$

In both cases, the lemma is proved. □

Proposition 3.7 *Let $p>4$. Then for every sequence (α_ε) converging to α, we have:*

$$\lim_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) = I_p(\alpha, h)\,.$$

Proof. Assume (without loss of generality) $\alpha_\varepsilon > 0$. Let us prove first that

$$\liminf_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \ge I_p(\alpha, h)\,. \tag{3.11}$$

Let v_ε be a solution of (3.4) given by Lemma 3.1. Set $\varphi_\varepsilon := \varepsilon\, v_\varepsilon$ and $[a_\varepsilon, b_\varepsilon] := \varphi_\varepsilon([-\frac{h}{2}, 0])$ (recall that φ_ε is even and non decreasing on $[-\frac{h}{2}, 0]$). Since the integral of φ_ε on $[-\frac{h}{2}, 0]$ tends to 0 with ε, the minimal value of φ_ε also tends to 0, so

$$\lim_{\varepsilon\to 0} a_\varepsilon = 0\,.$$

Euler equation leads to:

$$\varepsilon^4\, {\varphi'_\varepsilon}^2 = W(\varphi_\varepsilon) - C_\varepsilon\, \varphi_\varepsilon - D_\varepsilon \quad \text{ on } [-\tfrac{h}{2}, \tfrac{h}{2}],$$

where C_ε is the Lagrange multiplier associated to the integral constraint, and D_ε is an integration constant. The function $W(t) - C_\varepsilon\, t - D_\varepsilon$ is non negative on $[a_\varepsilon, b_\varepsilon]$, and vanishes for $t \in \{a_\varepsilon, b_\varepsilon\}$. Then there holds:

$$0 \leq C_\varepsilon = \frac{W(b_\varepsilon) - W(a_\varepsilon)}{b_\varepsilon - a_\varepsilon} \leq W'(a_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

According to the shape of W, only two behaviours of b_ε are possible.

First case: $\lim\limits_{\varepsilon\to 0} b_\varepsilon = 0$. By Schwartz's inequality, we have:

$$\begin{aligned}
\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) &\geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi_\varepsilon^2}{1+\varphi_\varepsilon^p}\, dx \\
&\geq \liminf_{\varepsilon\to 0} \frac{h}{1+b_\varepsilon^p}\, \frac{1}{\varepsilon^2} \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \varphi_\varepsilon dx \right)^2 \\
&\geq \frac{\alpha^2}{h}\cdot
\end{aligned} \tag{3.12}$$

Second case: $\lim\limits_{\varepsilon\to 0} b_\varepsilon = +\infty$. Recalling that φ_ε is even, we have :

$$\begin{aligned}
\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) &\geq \liminf_{\varepsilon\to 0} \int_{-\frac{h}{2}}^{\frac{h}{2}} 2\sqrt{W(\varphi_\varepsilon)}|\varphi_\varepsilon'|\, dx \\
&\geq \liminf_{\varepsilon\to 0} 4 \int_{a_\varepsilon}^{b_\varepsilon} \sqrt{W(t)} dt \\
&\geq 4 \int_0^{+\infty} \sqrt{W(t)} dt = k_p.
\end{aligned} \tag{3.13}$$

In both cases, inequality (3.11) holds.

It remains to show the upper-bound inequality:

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \leq I_p(\alpha, h)\,. \tag{3.14}$$

Consider the piecewise affine function v_ε defined on $[-\frac{h}{2}, \frac{h}{2}]$ by:

$$v_\varepsilon(x) := \frac{\alpha_\varepsilon}{h - \varepsilon^2} \inf \left\{ 1\,,\ \frac{1}{\varepsilon^2} \left(\tfrac{h}{2} - |x| \right) \right\}.$$

It satisfies $\displaystyle\int_{-\frac{h}{2}}^{\frac{h}{2}} v_\varepsilon\, dx = \alpha_\varepsilon$ and therefore we have

$$I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \leq G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) \leq \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(v_\varepsilon^2 + \varepsilon^4 v_\varepsilon'^2 \right) dx = \frac{\alpha_\varepsilon^2}{(h-\varepsilon^2)^2} \left(h + \frac{2}{3}\varepsilon^2 \right)\,.$$

It follows

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \le \frac{\alpha^2}{h}\cdot \tag{3.15}$$

Thus, by (3.15) and (3.14), the proposition will hold provided we prove the following claim

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \le \ k_p \,=\, 4\int_0^{+\infty} \sqrt{W}(t)dt. \tag{3.16}$$

Let γ such that $\dfrac{1}{p-2} < \gamma < \dfrac{1}{2}$. Then the differential equation

$$y' = (W(y))^\gamma \quad , \quad y(0) = 0 \ ,$$

has a unique monotone increasing solution φ on $[0, +\infty)$.

Moreover, noticing that for all $x \ge 0$ we have $x = \displaystyle\int_0^{\varphi(x)} W^{-\gamma}(t)\, dt$, we deduce that

$$\int_0^x \varphi(t)\, dt = \int_0^{\varphi(x)} \frac{u}{W^\gamma(u)} du \ge \int_1^{\varphi(x)} W^{-\gamma}(u)\, du = x - \int_0^1 W^{-\gamma}(u)\, du \ . \tag{3.17}$$

Then let $v_\varepsilon \in H^1_0(-\frac{h}{2}, \frac{h}{2})$ be the even function defined by

$$v_\varepsilon(x) := \varepsilon^{-1}\, \varphi\left(\frac{\frac{h}{2} - |x|}{\varepsilon^2} \right) .$$

From the estimate (3.17), we deduce that

$$\lim_{\varepsilon\to 0} \int_{-\frac{h}{2}}^{\frac{h}{2}} v_\varepsilon dx = \lim_{\varepsilon\to 0} 2\varepsilon \int_0^{\frac{h}{2\varepsilon^2}} \varphi(t)\, dt = +\infty \quad , \quad \lim_{\varepsilon\to 0} \varphi\left(\frac{h}{2\varepsilon^2}\right) = +\infty \ .$$

Therefore, by using the monotonicity of the map $\alpha \mapsto I_\varepsilon(\alpha, h, 0, 0)$ (Lemma 3.4)

$$\begin{aligned}
\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) &\le \limsup_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) \\
&= 2 \lim_{\varepsilon\to 0} \int_0^{\frac{h}{2\varepsilon^2}} \left(W(\varphi) + \varphi'^2\right) dx \\
&= 2 \int_0^{+\infty} \left(W(\varphi) + \varphi'^2\right) dx \\
&= 2 \int_0^{+\infty} \left(W^{1-\gamma} + W^\gamma\right)(t)\, dt \ .
\end{aligned}$$

Noticing that $\displaystyle\int_0^{+\infty} W^r(t)dt < +\infty$ for all $r > \dfrac{1}{p-2}$, we may pass to the limit as $\gamma \nearrow \frac{1}{2}$ in the right-hand side of the previous inequality. The claim (3.16) follows by dominated convergence. □

The case $1 < p < 4$ is more intricate. We will need the following lemma:

Lemma 3.8 *For all $\gamma > 0$, the minimization problem*

$$I_0(\gamma) := \inf_{w \in H^1(\mathbb{R})} \left\{ \int_{\{w>0\}} (w^{2-p} + w'^2)\, dx \ ; \ \int_{\mathbb{R}} w\, dy = \gamma \right\} \tag{3.18}$$

has a solution w_γ with compact support, and we have $I_0(\gamma) = k_p\, \gamma^{\frac{4-p}{2+p}}$.

Proof. For every $w \in H^1(\mathbb{R})$, we set

$$E(w) := \int_{\{w>0\}} (w^{2-p} + w'^2)\, dx\ .$$

Setting $v(x) := \gamma^{\frac{p}{2+p}-1} w\Big(x\ \gamma^{\frac{p}{2+p}}\Big)$, we have $E(w) = \gamma^{\frac{4-p}{2+p}} E(v)$ and $\int_{\mathbb{R}} v\, dx = 1$, so

$$I_0(\gamma) = \gamma^{\frac{4-p}{2+p}}\, I_0(1). \tag{3.19}$$

Let w_n be a minimizing sequence of $I_0(1)$. By rearrangement, we can choose w_n even and non increasing on $\mathbb{R}_+$. We denote by $M_n := w_n(0)$ its maximum value. We have

$$\begin{aligned} E(w_n) &= M_n^{1-p} + \int_{\{w_n>0\}} \left(w_n^{2-p} - M_n^{1-p}\, w_n + w_n'^2\right) dx \\ &\geq M_n^{1-p} + 2\int_{\{w_n>0\}} \sqrt{w_n^{2-p} - M_n^{1-p}\, w_n}\ |w_n'| dx \\ &\geq M_n^{1-p} + 4\int_0^{M_n} \sqrt{t^{2-p} - M_n^{1-p}\, t}\ dt \\ &\geq M_n^{1-p} + \frac{8}{p-1}\, M_n^{\frac{4-p}{2}} \int_0^\infty u^2 (u^2+1)^{-\frac{2p+1}{2p-2}} du. \end{aligned} \tag{3.20}$$

Let us define

$$\kappa(\lambda) := \lambda^{1-p} + \lambda^{\frac{4-p}{2}} \frac{8}{p-1} \int_0^\infty u^2 (u^2+1)^{-\frac{2p+1}{2p-2}} du \tag{3.21}$$

By taking the limit as n tends to $+\infty$ in (3.20), we have:

$$I_0(1) \geq \liminf_{n\to\infty} \kappa(M_n) \geq \inf_{\lambda} \Big\{\kappa(\lambda);\ \lambda \in \mathbb{R}_+\Big\}. \tag{3.22}$$

The function κ defined in (3.21) achieves its minimum at:

$$\overline{\lambda} = \left(\frac{p-1}{4} \left(\int_0^\infty (1+u^2)^{-\frac{2p+1}{2p-2}} du \right)^{-1} \right)^{\frac{2}{2+p}}$$

and the value of this minimum is given by

$$\kappa(\overline{\lambda}) \ = \ \frac{2+p}{4-p} \left(\frac{4}{p-1} \int_0^\infty (1+u^2)^{-\frac{2p+1}{2p-2}} du \right)^{\frac{2p-2}{2+p}}$$

$$
\begin{aligned}
&= \left(\frac{2+p}{4-p}\right)\left(\frac{4}{p-1}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{2p+1}{2p-2}-\frac{3}{2}}dt\right)^{\frac{2p-2}{2+p}} \\
&= \left(\frac{2+p}{4-p}\right)\left(\frac{2}{p-1}\,\frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{2p+1}{2p-2}-\frac{1}{2}\right)}{\Gamma\left(\frac{2p+1}{2p-2}\right)}\right)^{\frac{2p-2}{2+p}} \\
&= k_p\,. \qquad (3.23)
\end{aligned}
$$

Collecting (3.19), (3.22) and (3.23), we obtain

$$
I_0(\gamma) \;\geq\; k_p\,\gamma^{\frac{4-p}{2+p}}. \qquad (3.24)
$$

Now we consider the function $g(y) := \displaystyle\int_y^1 \frac{dt}{\sqrt{t^{2-p}-t}}$ which is continuous monotone decreasing on [0,1]. Setting $M := g(0)$, we denote by φ_0 the even function, compactly supported in $[-M, M]$, which coincides with the inverse of g on [0,M]. Clearly φ_0 satisfies the differential equation

$$
(\varphi_0')^2 \;=\; \varphi_0^{2-p} - \varphi_0 \qquad \text{on} \quad (-M, M)\,. \qquad (3.25)
$$

Then, by using the change of variable $u = \sqrt{t^{1-p}-1}$ and taking into account (3.23),(3.25), we obtain after some computation

$$
\begin{aligned}
m_0 := \int_{-\infty}^{+\infty}\varphi_0 &= 2\int_0^1 \frac{tdt}{\sqrt{t^{2-p}-t}} = \frac{4}{p-1}\int_0^\infty (1+u^2)^{-\frac{2p+1}{2p-2}}du \\
&= \left(\tfrac{4-p}{2+p}\,k_p\right)^{\frac{2+p}{2-p}}, \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
E(\varphi_0) = 2\int_0^1 \frac{2t^{2-p}-t}{\sqrt{t^{2-p}-t}}\,dt &= 4\int_0^1 \sqrt{t^{2-p}-t}\,dt + 2\int_0^1 \frac{tdt}{\sqrt{t^{2-p}-t}} \\
&= \frac{p+2}{4-p}\,m_0\,. \qquad (3.27)
\end{aligned}
$$

Eventually we find an admissible function for (3.18) by setting:

$$
w_\gamma(x) := (m_0\,\gamma^{-1})^{-\frac{2}{2+p}}\,\varphi_0\Big((m_0\,\gamma^{-1})^{\frac{p}{2+p}}\,x\Big).
$$

Clearly w_γ is compactly supported and by (3.26),(3.27), its energy satisfies

$$
E(w_\gamma) = \left(\frac{\gamma}{m_0}\right)^{\frac{4-p}{2+p}} E(\varphi_0) = k_p\,\gamma^{\frac{4-p}{2+p}}. \qquad (3.28)
$$

The optimality of w_γ follows from (3.24) and (3.28). □

Proposition 3.9 *Let p such that $1 < p < 4$. Then for every sequence (α_ε) converging to α, we have:*

$$
\lim_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) = I_p(\alpha, h)\,.
$$

Proof. Assume (without loss of generality) $\alpha_\varepsilon > 0$. First, we prove

$$\liminf_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \geq I_p(\alpha, h)\,. \tag{3.29}$$

We may assume that the left-hand side of (3.29) is finite and, possibly passing to a subsequence, that it is actually a limit. Let v_ε be a solution of problem (3.4) given by Lemma 3.1. and denote by M an upper bound of the energy:

$$G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) \leq M. \tag{3.30}$$

Setting $w_\varepsilon(x) := \varepsilon^{p/p-1}\ v_\varepsilon\left(x\,\varepsilon^{p/p-1}\right)$, we get

$$G_\varepsilon(v_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})) = \int_{|x|<\frac{h}{2\varepsilon^{p/p-1}}} \left(\frac{w_\varepsilon^2}{\varepsilon^{p/p-1} + w_\varepsilon^p} + w_\varepsilon'^2\right) dx\ , \tag{3.31}$$

where w_ε satisfies

$$\int_{|x|<\frac{h}{2\varepsilon^{p/p-1}}} w_\varepsilon\, dx = \alpha_\varepsilon\ , \quad w_\varepsilon\Big(\pm\frac{h}{2\varepsilon^{p/p-1}}\Big) = 0. \tag{3.32}$$

Now, we fix some $\eta > 0$ and set $\beta_\varepsilon := \int_{\{w_\varepsilon<\eta\,\varepsilon^{\frac{1}{p-1}}\}} w_\varepsilon\, dx$. Possibly passing to a subsequence, we may assume that:

$$\beta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \beta\ , \quad \beta \in [0, \alpha]. \tag{3.33}$$

First, we claim that (w_ε) is weakly precompact in $H^1(\mathbb{R})$ and that any cluster point w satisfies

$$\int_{\mathbb{R}} w\, dx = \alpha - \beta. \tag{3.34}$$

Indeed, by (3.30), (3.31) and (3.32), we have for a suitable constant C

$$\|w_\varepsilon\|_{H^1(\mathbb{R})} = \|w_\varepsilon\|_{L^2(\mathbb{R})} + \|w_\varepsilon'\|_{L^2(\mathbb{R})} \leq \|w_\varepsilon\|_{L^1(\mathbb{R})} + 2\|w_\varepsilon'\|_{L^2(\mathbb{R})} \leq C.$$

Therefore, (w_ε) is weakly precompact in $H^1(\mathbb{R})$ and we may assume that it converges to w in $L^1_{\text{loc}}(\mathbb{R})$. Since (w_ε) is non increasing on $\mathbb{R}_+$, so is w and for any $\delta > 0$, the set $\{w_\varepsilon > \delta\}$ is then a bounded interval, so that:

$$\lim_{\varepsilon\to 0} \int_{\{w_\varepsilon>\delta\}} w_\varepsilon\, dx = \int_{\{w>\delta\}} w\, dx. \tag{3.35}$$

By (3.32) and (3.33), we deduce

$$\int_{\{w>\delta\}} w\, dx = \alpha - \beta - \lim_{\varepsilon\to 0} \int_{A^\delta_{\varepsilon,\eta}} w_\varepsilon\, dx\ , \tag{3.36}$$

where $A^\delta_{\varepsilon,\eta} := \{\eta\,\varepsilon^{\frac{1}{p-1}} < w_\varepsilon < \delta\}$.

By (3.30) we have $\int_{A^\delta_{\varepsilon,\eta}} w_\varepsilon^2\,dx \le M(\delta^p + \varepsilon^{\frac{p}{p-1}})$ and $\int_{A^\delta_{\varepsilon,\eta}} w_\varepsilon^{2-p}\,dx \le M(1+\eta^{-p})$, so that by Hölder's inequality:

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{A^\delta_{\varepsilon,\eta}} w_\varepsilon\,dx &\le \limsup_{\varepsilon\to 0}\left(\left(\int_{A^\delta_{\varepsilon,\eta}} w_\varepsilon^{2-p}\,dx\right)^{\frac{1}{p}}\left(\int_{A^\delta_{\varepsilon,\eta}} w_\varepsilon^{2}\,dx\right)^{1-\frac{1}{p}}\right)\\ &\le M\,\delta\,(1+\eta^{-p})^{1-\frac{1}{p}}.\end{aligned}$$

The claim follows by letting δ tend to 0 in (3.36).

Let us now minorize the energy of v_ε. By (3.31), we have

$$G_\varepsilon(v_\varepsilon,(-\tfrac{h}{2},\tfrac{h}{2})) \ge \underbrace{\int_{\{w_\varepsilon<\eta\,\varepsilon^{\frac{1}{p-1}}\}}\left(\frac{w_\varepsilon^2}{\varepsilon^{p/p-1}+w_\varepsilon^p}\right)dx}_{I_1^\varepsilon} + \underbrace{\int_{\{w_\varepsilon>\eta\,\varepsilon^{\frac{1}{p-1}}\}}\left(\frac{w_\varepsilon^2}{\varepsilon^{p/p-1}+w_\varepsilon^p}+w_\varepsilon'^2\right)dx}_{I_2^\varepsilon}$$

By Schwartz's inequality and (3.33):

$$\begin{aligned}\liminf_{\varepsilon\to 0} I_1^\varepsilon &\ge \frac{1}{1+\eta^p}\,\liminf_{\varepsilon\to 0}\int_{\{w_\varepsilon<\eta\,\varepsilon^{\frac{1}{p-1}}\}} w_\varepsilon^2\,dx\\ &\ge \frac{1}{1+\eta^p}\,\liminf_{\varepsilon\to 0}\left[\frac{\beta_\varepsilon^2}{\left|\{w_\varepsilon<\eta\,\varepsilon^{\frac{1}{p-1}}\}\right|}\right] \ge \frac{1}{1+\eta^p}\,\frac{\beta^2}{h}.\end{aligned}\tag{3.37}$$

Since w_ε weakly converges to w, by Fatou's lemma, (3.34), and Lemma 3.8, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0} I_2^\varepsilon &\ge \int_{\mathbb{R}}\liminf_{\varepsilon\to 0}\left[\mathbf{1}_{\{w_\varepsilon>\eta\,\varepsilon^{\frac{1}{p-1}}\}}\left(\frac{w_\varepsilon^2}{\varepsilon^{p/p-1}+w_\varepsilon^p}+w_\varepsilon'^2\right)\right]dx\\ &\ge \inf_{w\in H^1(\mathbb{R})}\left\{\int_{\{w>0\}}(w^{2-p}+w'^2)\,dy\ ;\ \int_{\mathbb{R}} w\,dy=\alpha-\beta\right\}\\ &\ge k_p\,(\alpha-\beta)^{\frac{4-p}{2+p}}\end{aligned}\tag{3.38}$$

(here we have used that, by the pointwise convergence of w_ε, we have the inequality $\liminf_{\varepsilon\to 0}\mathbf{1}_{\{w_\varepsilon>\eta\,\varepsilon^{\frac{1}{p-1}}\}} \ge \mathbf{1}_{\{w>0\}}$ *a.e.*).

Eventually, collecting (3.37), (3.38), and taking the limit $\eta\to 0$, we get (3.29).

Now let us prove that

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon,h,0,0) \le I_p(\alpha,h).\tag{3.39}$$

Let $\beta \in [0,\alpha]$, and $\gamma > \alpha - \beta$. We consider the solution w_γ of (3.18), with compact support, given by Lemma 3.8, and construct for any $\varepsilon > 0$ the function

$$\begin{cases} w_\varepsilon(x) := \sup\left\{\dfrac{\beta}{h}\ ;\ \varepsilon^{\frac{-p}{p-1}}\ w_\gamma\Big(\varepsilon^{\frac{-p}{p-1}}\ x\Big)\right\} \ \text{on } [-\frac{h}{2}+\varepsilon^2, \frac{h}{2}-\varepsilon^2] \\ w_\varepsilon(x) \text{ affine on } [-\frac{h}{2}, -\frac{h}{2}+\varepsilon^2] \cup [\frac{h}{2}-\varepsilon^2, \frac{h}{2}]\ ,\ w_\varepsilon(\pm\frac{h}{2}) = 0. \end{cases}$$

It is clear that $w_\varepsilon \rightharpoonup \dfrac{\beta}{h}\, dx + \gamma\, \delta_0(dx)$ (in the sense of measures). Hence $\displaystyle\int_{-\frac{h}{2}}^{\frac{h}{2}} w_\varepsilon\, dx$ tends to $\beta+\gamma$, and using the monotony property proved in Lemma 3.4, we obtain

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \leq \limsup_{\varepsilon\to 0} G_\varepsilon(w_\varepsilon, (-\tfrac{h}{2}, \tfrac{h}{2})). \tag{3.40}$$

We have

$$\begin{aligned} G_\varepsilon(w_\varepsilon, \{w_\varepsilon < \beta/h\}) &\leq 2\varepsilon^2\left(\frac{\beta^2}{h^2} + \varepsilon^{\frac{3p}{p-1}}\left(\frac{\beta}{\varepsilon^2 h}\right)^2\right) \underset{\varepsilon\to 0}{\longrightarrow} 0\ ,\\ G_\varepsilon(w_\varepsilon, \{w_\varepsilon = \beta/h\}) &\leq h\,\frac{\beta^2}{h^2} = \frac{\beta^2}{h}\ ,\\ G_\varepsilon(w_\varepsilon, \{w_\varepsilon > \beta/h\}) &\leq E(w_\gamma) = k_p\ \gamma^{\frac{4-p}{2+p}}. \end{aligned}$$

Then (3.40) becomes

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, 0, 0) \leq \frac{\beta^2}{h} + k_p\ \gamma^{\frac{4-p}{2+p}},$$

which holds whenever $\gamma+\beta > \alpha$, leading to (3.39). □

Proposition 3.10 *Let $\nu := 1$ if $p > 4$, $\nu := \frac{p}{p-1}$ if $1 < p < 4$. Let α_ε, r_ε and s_ε verifying*

$$\alpha_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \alpha \quad , \quad \varepsilon^\nu\, r_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad , \quad \varepsilon^\nu\, s_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Then

$$\lim_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon) = I_p(\alpha, h). \tag{3.41}$$

Proof. Let h_ε, δ_ε (*resp.* h'_ε, δ'_ε) be associated to the sequence r_ε (*resp.* s_ε) through Lemma 3.6. Then passing to the limit in ε in the following inequalities (deduced from (3.8) in Lemma 3.3)

$$I_\varepsilon(\alpha_\varepsilon+\delta_\varepsilon+\delta'_\varepsilon, h+h_\varepsilon+h'_\varepsilon, 0, 0) \leq I_\varepsilon(\delta_\varepsilon, h_\varepsilon, 0, r_\varepsilon) + I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon) + I_\varepsilon(\delta'_\varepsilon, h'_\varepsilon, s_\varepsilon, 0),$$
$$I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon) \leq I_\varepsilon(\delta_\varepsilon, h_\varepsilon, r_\varepsilon, 0) + I_\varepsilon(\alpha_\varepsilon-\delta_\varepsilon-\delta'_\varepsilon, h-h_\varepsilon-h'_\varepsilon, 0, 0) + I_\varepsilon(\delta'_\varepsilon, h'_\varepsilon, 0, s_\varepsilon),$$

we get

$$\begin{aligned} \liminf_{\varepsilon\to 0}\ I_\varepsilon(\alpha_\varepsilon+\delta_\varepsilon+\delta'_\varepsilon, h+h_\varepsilon+h'_\varepsilon, 0, 0) &\leq \liminf_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon)\ ,\\ \limsup_{\varepsilon\to 0}\ I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon) &\leq \limsup_{\varepsilon\to 0}\ I_\varepsilon(\alpha_\varepsilon-\delta_\varepsilon-\delta'_\varepsilon, h-h_\varepsilon-h'_\varepsilon, 0, 0). \end{aligned}$$

Let $\eta > 0$. By the monotonicity property stated in Lemma 3.5, and by applying the Proposition 3.7 in the case $p > 4$ (or the Proposition 3.9 in the case $p \in (1,4)$), we are led to

$$I_p(\alpha, h+\eta) \leq \liminf_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon + \delta_\varepsilon + \delta'_\varepsilon, h+\eta, 0, 0) \leq \liminf_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon) \leq$$
$$\leq \limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon, h, r_\varepsilon, s_\varepsilon) \leq \limsup_{\varepsilon\to 0} I_\varepsilon(\alpha_\varepsilon - \delta_\varepsilon - \delta'_\varepsilon, h-\eta, 0, 0) \leq I_p(\alpha, h-\eta).$$

The conclusion follows by letting η tend to 0. □

Remark 3.11 *By repeating the arguments of the proof and by using the monotonicity property stated in Lemma 3.4, we obtain that, if $\alpha = \lim \alpha_\varepsilon = +\infty$, then (3.41) is replaced by*

$$\liminf_{\varepsilon\to 0} \; I_\varepsilon(\alpha_\varepsilon, h) \;\geq\; I_p(\infty, h) \;, \tag{3.42}$$

(where, for all $h > 0$, $I_p(\infty, h) := +\infty$ if $p < 4$ and $I_p(\infty, h) := k_p$ if $p > 4$).

4. Compactness

In this section, we consider a sequence (u_ε) which is bounded in $L^2(\Omega)$ and in energy. We denote by M a constant such that

$$F_\varepsilon(u_\varepsilon, \Omega) + \|u_\varepsilon\|_{L^2} \leq M \;. \tag{4.1}$$

Without loss of generality we may assume that u_ε converges weakly to u in L^2.

In order to use Proposition 3.10, we need first to prove

Lemma 4.1 *Under (4.1), the following strong convergences hold:*

(i) If $p \in (1,4)$, $\varepsilon^{\frac{p}{p-1}}\, u'_\varepsilon \to 0$ in $L^2(\Omega)$

(ii) If $p > 4$, $\varepsilon\, u'_\varepsilon \to 0$ in $L^1(\Omega)$ (in $L^q(\Omega)$ for $q \in [1,2)$).

Proof. Recall that by Nirenberg inequality,[20] there exists $C > 0$ such that for all $u \in W^{2,2}(\Omega)$

$$\|u'\|_{L^2(\Omega)} \leq C\Big(\|u''\|^{\frac12}_{L^2(\Omega)} \|u\|^{\frac12}_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \Big). \tag{4.2}$$

If $p < 4$, the upper bound (4.1) yields

$$\|u''_\varepsilon\|_{L^2(\Omega)} \leq M^{\frac12}\, \varepsilon^{\frac{-3p}{2(p-1)}},$$

and using (4.2), we get:

$$\|\varepsilon^{\frac{p}{p-1}}\, u'_\varepsilon\|_{L^2(\Omega)} \leq C\, \varepsilon^{\frac{p}{4(p-1)}}.$$

If $p > 4$, by (4.1), we have

$$\varepsilon^4 \int_\Omega u''^2_\varepsilon \, dx \leq M \;, \quad \int_\Omega W(\varepsilon\, u'_\varepsilon)\, dx \leq M\, \varepsilon^2 \;.$$

Therefore, by (4.2), the sequence $\varphi_\varepsilon := \varepsilon u'_\varepsilon$ is bounded in $L^2(\Omega)$ and, taking into account that $\varepsilon u_\varepsilon$ tends to 0 in $L^2(\Omega)$, we obtain

$$\varphi_\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega) \quad , \quad W(\varphi_\varepsilon) \to 0 \quad \text{strongly in } L^1(\Omega) \tag{4.3}$$

Let us denote, for every $t > \eta$, $k(\eta, t)$ the minimum value of W over $[\eta, t]$. Then $k(\eta, t) > 0$ and we have for a suitable constant $C > 0$:

$$\begin{aligned} |\{\varphi_\varepsilon > \eta\}| \quad &\leq |\{\eta < \varphi_\varepsilon \leq t\}| + \frac{1}{t^2} \int |\varphi_\varepsilon|^2 dx \\ &\leq \; k(\eta, t) \int W(\varphi_\varepsilon)\, dx \; + \; \frac{C}{t^2} \;. \end{aligned}$$

Thus (4.3) yields for every $t > 0$:

$$\limsup_{\varepsilon \to 0} \; |\{\varphi_\varepsilon > \eta\}| \; \leq \frac{C}{t^2} \;.$$

The convergence in measure of φ_ε to 0 follows by letting $t \to \infty$ and then assertion ii) is proved by using Vitali's convergence theorem. □

In the sequel we will need some specific notations: for any interval $J \subset \Omega$ and for any $x \in \Omega$ we define

$$\omega_\varepsilon(J) = \sup_{x_1, x_2 \in J} |u_\varepsilon(x_1) - u_\varepsilon(x_2)| \quad , \quad m_\varepsilon(J) = \int_J |u'_\varepsilon|\, dx$$

$$\omega(J) = \limsup_{\varepsilon \to 0} \omega_\varepsilon(J) \quad , \quad m(J) = \limsup_{\varepsilon \to 0} m_\varepsilon(J) \;.$$

The asymptotic oscillation of the sequence u_ε is represented locally at every $x \in \Omega$ by

$$\omega_0(x) = \lim_{\delta \to 0^+} \omega(J_\delta(x)) \qquad (\text{where } J_\delta(x) := \{y \in \Omega; |y - x| < \frac{\delta}{2}\}) \;.$$

We set:

$$D \; := \; \{x \in \Omega \; ; \; \omega_0(x) > 0\} \;. \tag{4.4}$$

By (4.1) the sequence of energy densities $\mu_\varepsilon := F_\varepsilon(u_\varepsilon, \cdot)$ is bounded in $L^1(\Omega)$, and therefore admits a weak-star limit in the space of bounded measures on Ω denoted by μ. From now on, we select from the original sequence (u_ε) a subsequence (still denoted with the same symbol) and a *countable dense* subset N of Ω such that the following properties are satisfied

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly-star} \;, \quad \mu(N) = 0 \tag{4.5}$$

$$m_\varepsilon(J) \to m(J) \quad (\in [0, +\infty]) \quad \text{whenever } \partial J \subset N \tag{4.6}$$

$$\varepsilon^\nu \; u'_\varepsilon(x) \to 0 \;, \; \sup_\varepsilon |u_\varepsilon(x)| < +\infty \quad \text{for all } x \in N, \tag{4.7}$$

where in (4.7), the exponent ν is defined, according to Lemma 4.1, by $\nu = 1$ if $p > 4$ and $\nu = \frac{p}{p-1}$ if $p \in (1, 4)$.

We notice that $\omega_\varepsilon(J) \leq m_\varepsilon(J)$ and therefore $\omega(J) \leq m(J)$.

Lemma 4.2 *For all J such that $\partial J \subset N$, we have:*

$$\mu(J) \ \geq \ I_p(m(J), |J|) \ \geq \ I_p(\omega(J), |J|) \ .$$

Proof. The function $\widetilde{u}_\varepsilon$ defined by

$$\widetilde{u}_\varepsilon(x) := \int_{J \cap \{s<x\}} |u'_\varepsilon(s)|\, ds \ ,$$

is monotone on J, increasing from 0 to $m_\varepsilon(J)$, and satisfies $|\widetilde{u}'_\varepsilon| = |u'_\varepsilon|$ and $|\widetilde{u}''_\varepsilon| = |u''_\varepsilon|$ a.e. Therefore, denoting by $r_\varepsilon(J), s_\varepsilon(J)$ the traces of $|u'_\varepsilon|$ on ∂J, it holds:

$$\mu_\varepsilon(J) \ = \ F_\varepsilon(u_\varepsilon, J) \ = \ F_\varepsilon(\widetilde{u}_\varepsilon, J) \ \geq I_\varepsilon\Big(m_\varepsilon(J), |J|, r_\varepsilon(J), s_\varepsilon(J)\Big) \ .$$

As $\partial J \subset N$, taking into account (4.5),(4.6) (4.7) and (3.41) ((3.42) if $m(J) = +\infty$), we deduce that

$$\mu(J) = \lim_{\varepsilon\to 0} \mu_\varepsilon(J) \geq \liminf_{\varepsilon\to 0} I_\varepsilon(m_\varepsilon(J), |J|, r_\varepsilon(J), s_\varepsilon(J)) \geq I_p(m(J), |J|) \ .$$

As $m(J) \geq \omega(J)$, it is clear that $I_p(m(J), |J|) \geq I_p(\omega(J), |J|)$. □

Lemma 4.3 *The set D defined by (4.4) is at most countable for all $p > 1$. It is finite if $p > 4$.*

Proof. Let us define the sets $D_q := \{x \mid \omega_0(x) > 1/q\}$ so that $D := \cup_{q\in\mathbb{N}} D_q$. By Lemma 4.2, for all $x \in D_q$ we can select a suitable sequence $\delta \to 0$, such that

$$\begin{aligned}\mu(x) \ &= \ \lim_\delta \mu(J_\delta(x)) \geq \liminf_\delta I_p(\omega(J_\delta(x)), \delta) \\ &\geq \ \liminf_\delta I_p(\omega_0(x), \delta) \geq I_p\left(\frac{1}{q}, 0^+\right) = k_p \left(\frac{1}{q}\right)^{\gamma_p} \ .\end{aligned}$$

Hence, for all $q \in \mathbb{N}$

$$\mu(D) \geq \mu(D_q) \geq \ k_p \left(\frac{1}{q}\right)^{\gamma_p} \ \sharp\,(D_q) \ .$$

As μ has a finite total mass, D_q is finite and D is at most countable. Moreover, if $p > 4$, $\gamma_p = 0$ and $\sharp\,(D) \leq \frac{M}{k_p}$. □

Lemma 4.4 *On every compact $K \subset \Omega\backslash D$, the sequence (u_ε) converges uniformly to u.*

Proof. For all $x \in K$, the condition $\omega_0(x) = 0$ implies that (u_ε) is equi-continuous at x and by (4.7), the sequence $(u_\varepsilon(x))$ is bounded in $\mathbb{R}$ for all x in the dense subset N. The conclusion follows by Ascoli's Theorem. □

Lemma 4.5 *If $1 < p < 4$, the sequence (u_ε) is bounded in $W^{1,1}(\Omega)$.*

Proof. We denote $\Omega = (a,b)$, and we take $\eta \in (0, \frac{b-a}{2})$. We extend u_ε on the set $\Omega_\eta := (a-\eta, b+\eta)$ by setting for all $t \in (0,\eta)$:

$$u_\varepsilon(a-t) := 2u_\varepsilon(a) - u_\varepsilon(a+t)\,, \quad u_\varepsilon(b+t) := 2u_\varepsilon(b) - u_\varepsilon(a-t).$$

It is easy to check that $\sup_\varepsilon F_\varepsilon(u_\varepsilon, \Omega_\eta) < +\infty$. Then repeating all arguments above replacing Ω by Ω_η and μ by the weak limit of $F_\varepsilon(u_\varepsilon, \cdot)$ on Ω_η, we deduce from Lemma 4.2 that

$$\limsup_\varepsilon \int_J |u'_\varepsilon| = \limsup_\varepsilon m_\varepsilon(J) \;\le \mu(J) < +\infty$$

holds at least for one interval J such that $J \supset \Omega$. □

Proof of part i) of Theorem 2.1. The case $p < 4$ is a direct consequence of Lemma 4.5 and of the compact imbedding from $W^{1,1}(\Omega)$ in $L^q(\Omega)$ $(q < +\infty)$.

In the case $p > 4$, we know from Lemma 4.3 and Lemma 4.4 that we can select a subsequence of (u_ε) converging weakly to u in $L^2(\Omega)$ and such that (u_ε) converges uiformly to u on the complementary of some finite set D. This implies a.e. convergence on Ω and the conclusion follows by using Vitali's convergence Theorem. □

5. Proof of assertions ii) and iii) of Theorem 2.1

5.1. *Lower-bound inequality.*

Let (u_ε) be a sequence with bounded energy and weakly converging to u in $L^2(\Omega)$. In the same way as in section 4, we select a subsequence still denoted (u_ε) and a dense countable subset of Ω such that (4.5)(4.6)(4.7) are satisfied. We may also impose the following conditions

$$\liminf_{\varepsilon\to 0} F(u_\varepsilon, \Omega) \;= \lim_{\varepsilon\to 0} \;\mu_\varepsilon(\Omega) \tag{5.1}$$

$$D \cap N = \emptyset \qquad (D \text{ defined by } (4.4)) \tag{5.2}$$

By (4.5), assertion ii) of Theorem 2.1 will be proved if we show that

$$\mu(J) \;\ge F(u,J) \qquad \text{for all interval } \; J \subset \Omega \; \text{ such that } \partial J \subset N\;. \tag{5.3}$$

Indeed, as μ is bounded, it will follow that $F(u,\cdot)$ (see definition (1.2)) is a Radon measure on Ω and (5.3) extends to $J = \Omega$ by using inner regularity.

We first prove (5.3) assuming that u belongs to $L^\infty(J)$. The main ingredients are the Lemma 4.2 and the lower semicontinuity properties of the functional $F(\cdot, J)$ (see [4,13]).

Lemma 5.1 *Let $J := (a_0, b_0) \subset \Omega$ such that $a_0, b_0 \in N$ and assume that $u \in L^\infty(J)$. Then $\mu(J) \ge F(u,J)$.*

To establish (5.3) in the general case, it is enough to check that u belongs to $L^\infty(\Omega)$. In the case $p \in (1,4)$, this is a direct consequence of the boundedness of (u_ε) in $W^{1,1}(\Omega)$ stated in Lemma 4.5. If $p > 4$, we know from Lemma 4.3 that D is a finite set. Then for every interval J whose closure is contained in $\Omega \setminus D$, (u_ε) converges uniformly (Lemma 4.4) and u is continuous. By Lemma 5.1 we deduce that $u \in H^1(J)$ and $\int_J u'^2 = F(u,J) \le \mu(J) \le \mu(\Omega) < +\infty$. The last upper-bound being independant of J, it follows that $u \in H^1(\Omega \setminus D)$ and u admits finite left-hand side and right-hand side limits at every point of D. Hence $u \in L^\infty(\Omega)$ (in fact $u \in SBV(\Omega)$) and the conclusion. □

Proof of Lemma 5.1. Let $\delta > 0$ and $B_j := (a_j, a_{j+1})$ a subdivision of J such that $|B_j| < \delta$ and $a_j \in N$. Then by (5.2) and Lemma 4.4, there holds $u_\varepsilon(a_j) \to u(a_j)$. Hence

$$|u(a_{j+1}) - u(a_j)| \ = \ \lim_\varepsilon |u_\varepsilon(a_{j+1}) - u_\varepsilon(a_j)| \ \le \ \omega(B_j) \ .$$

By applying Lemma 4.2, we deduce

$$\mu(J) \ = \ \sum_j \ \mu(B_j) \ge \sum_j I_p\Big(u(a_{j+1}) - u(a_j), |B_j|\Big). \tag{5.4}$$

Now, according to Lemma 3.2, we may consider a monotone function u_j^δ such that

$$F(u_j^\delta, B_j) \ = \ I_p\Big(u(a_{j+1}) - u(a_j), |B_j|\Big) \ .$$

Let us consider the function u^δ defined on J by setting for $x \in B_j$: $u^\delta(x) := u_j^\delta\Big(x - \frac{a_j + a_{j+1}}{2}\Big) + u(a_j)$. By (5.4), we a led to

$$\mu(J) \ \ge \ \sum_j F(u^\delta, B_j) = \ F(u^\delta, J) \ . \tag{5.5}$$

Since by construction we have $\|u^\delta\|_{L^\infty(J)} \le \|u\|_{L^\infty(J)}$, then for every $x \in \mathcal{S}_{u^\delta} \cap J$

$$|(u^\delta)^+ - (u^\delta)^-| \le \Big(2\,\|u\|_{L^\infty(B)}\Big)^{1-\gamma_p} \ |(u^\delta)^+ - (u^\delta)^-|^{\gamma_p}.$$

Therefore it exists a constant $C > 0$ such that

$$\begin{aligned} F(u^\delta, J) \ &= \ \int_J |(u^\delta)'|^2\,dx + k_p \sum_{\mathcal{S}_{u^\delta} \cap J} \Big|(u^\delta)^+ - (u^\delta)^-\Big|^{\gamma_p} \\ &\ge \ C\Big(\int_B |(u^\delta)'|\,dx + \sum_{\mathcal{S}_{u^\delta} \cap J} |(u^\delta)^+ - (u^\delta)^-|\Big) = C \ \int_J |Du^\delta| \ . \end{aligned}$$

This together with (5.5) entails that (u^δ) is bounded in $BV(J)$ and converges strongly to u in $L^1(J)$. Then by the lower semicontinuity of F on the space $BV(J)$ (see Ambrosio[4]), we may conclude by passing to the limit as δ tends to 0 in (5.5). □

5.2. *Upper-bound inequality.*

Let $\Omega = (a,b)$ and u be a function of $SBV(\Omega)$. We denote by Du the measure representing the distributional gradient of u. Fix $\delta > 0$ and $B_j := (a_j, a_{j+1})$ a subdivision of Ω such that $|B_j| < \delta$ and $a_j \notin \mathcal{S}_u$. For every j, by Lemma 3.1, there exists a monotone function u_ε^j such that:

$$\begin{aligned} &u_\varepsilon^j(a_j) = u(a_j),\ u_\varepsilon^j(a_{j+1}) = u(a_{j+1}),\ (u_\varepsilon^j)'(a_j) = (u_\varepsilon^j)'(a_{j+1}) = 0, \\ &F_\varepsilon(u_\varepsilon^j, (a_j, a_{j+1})) = I_\varepsilon(u(a_{j+1}) - u(a_j), a_{j+1} - a_j, 0, 0). \end{aligned}$$

Then the function u_ε^δ defined on Ω by $u_\varepsilon^\delta(x) := u_\varepsilon^j(x)$ on each B_j belongs to $W^{2,2}(\Omega)$ and satisfies by construction

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(B_j)} \le \|u_\varepsilon\|_{L^\infty(\Omega)} \\ &F_\varepsilon(u_\varepsilon^\delta, \Omega) = \sum_j F_\varepsilon(u_\varepsilon^j, B_j) = \sum_j I_\varepsilon(u(a_{j+1}) - u(a_j), |B_j|, 0, 0). \end{aligned}$$

Using Propositions 3.7 or 3.9, we get:

$$\begin{aligned} \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon^\delta, \Omega) &\le \sum_j \limsup_{\varepsilon\to 0} I_\varepsilon(u(a_{j+1}) - u(a_j), |B_j|, 0, 0) \\ &\le \sum_j I_p(u(a_{j+1}) - u(a_j), |B_j|) \\ &\le \sum_j F(u, B_j) = F(u, \Omega). \end{aligned} \tag{5.6}$$

Moreover, since u_ε^j is monotone on (a_j, a_{j+1}), we have:

$$\int_{B_j} |(u_\varepsilon^j)'|\, dx = \left| \int_{B_j} (u_\varepsilon^j)'\, dx \right| = |u_\varepsilon^j(a_{j+1}) - u_\varepsilon^j(a_j)| = |u(a_{j+1}) - u(a_j)| \le \int_{B_j} |Du|.$$

Thus, applying Poincaré's inequality on each B_j, we obtain:

$$\begin{aligned} \int_\Omega |u_\varepsilon^\delta - u|\, dx &= \sum_j \int_{B_j} |u_\varepsilon^j - u|\, dx \\ &\le \delta \sum_j \left(\int_{B_j} |(u_\varepsilon^j)'| dx + \int_{B_j} |Du| \right) \le 2\delta \int_\Omega |Du|. \end{aligned}$$

Hence

$$\limsup_{\delta\to 0}\ \limsup_{\varepsilon\to 0} \int_\Omega |u_\varepsilon^\delta - u|\, dx = 0.$$

By a classical diagonalization argument,[5] we can choose a sequence δ_ε tending to 0 such that the sequence $u_\varepsilon := u_\varepsilon^{\delta_\varepsilon}$ converges to u strongly in $L^1(\Omega)$ and verifies:

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) \le F(u, \Omega).$$

The proof is completed noticing that, due to the L^∞ bound for u_ε, the convergence of u_ε holds also in $L^2(\Omega)$. □

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