Cahn and Hilliard fluid on an oscillating boundary

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Abstract. A Cahn and Hilliard fluid is in equilibrium in a solid container. The rugosity of the boundaries is taken into account through the assumption of an oscillating boundary whose period and amplitude are of the same order of magnitude as the thickness of the interface. We study the Γ-limit of this problem when this length tends to zero. We obtain a homogenized boundary energy and we show that rugosity may modify the wetting property of the wall until it is completely wet.

1.Introduction

In 1959 Cahn and Hilliard [1] wrote a continuum model for two-phase fluids. The postulated energy was of the form:

$$E(u) = \int_\Omega W(u)\ dx + \lambda \int_\Omega |Du|^2\ dx + \int_{\partial\Omega} \sigma(u)\ dH^{\mathrm{N}-1}$$

u is the density of the fluid. u is positive, $\int_\Omega u\ dx = m$. $W(u)$ is the volumic free energy for an homogeneous fluid of density u. It is a non convex function (for example the energy associated with the Van der Waals pressure). See figure 1 which defines α_1 and α_2, values of u in the phases. λ is a physical parameter which may be deduced from experimental surface tension as far as the function W is known. $\sigma(u)$ is a surfacic energy which characterizes the interactions between a fluid of density u and the wall ($\sigma(u)$ is positive). This model has its own length L^c :

$$L^c = \sqrt{\lambda}\frac{\alpha_2 - \alpha_1}{2\int_{\alpha_1}^{\alpha_2} \sqrt{W(u)}\ du}$$

which is characteristic of the thickness of the transition layer between the two phases $u = \alpha_1$ and $u = \alpha_2$.

As L^c is very small it is natural to study the asymptotic behaviour of the model as L^c tends to zero. This procedure is purely mathematical and the model does not include assumptions on the behaviour of the other physical quantities W, σ, m and Ω itself with respect to L^c. We emphasize that the postulated behaviour for these quantities are primordial for the resulting model.

If W, σ, m, Ω are constant, we lose every surface tension effect. For that the usual asymptotic problem is concerned with the limit as ε tends to zero of the *rescaled energy* :

$$E^{\varepsilon}(u) = \int_{\Omega} \frac{W_0(u)}{\varepsilon}\ dx + \varepsilon \int_{\Omega} |Du|^2\ dx + \int_{\partial\Omega} \sigma(u)\ dH^{\mathrm{N}-1}$$

where $W_0 \geq 0, W_0(\alpha_1) = W_0(\alpha_2) = 0$ ($W_0(u) = W(u) - l(u)$ see figure 1)

This problem was completely solved in 1987 by L. Modica [2]. The resulting energy is given by:

$$\begin{aligned} E_0(u) &= c\ H^{\mathrm{N}-1}(\partial^* A \cap \Omega) - \widehat{c}\ H^{\mathrm{N}-1}(\partial^* A \cap \partial\Omega) \\ &\qquad \text{if } u = \alpha_1\ 1_A + \alpha_2\ 1_{\Omega\setminus A} \ \text{ and } \ per_{\Omega}(A) < +\infty \\ E_0(u) &= +\infty \ \text{ otherwise,} \end{aligned}$$

where

$$\begin{aligned} c &= 2\int_{\alpha_1}^{\alpha_2} \sqrt{W_0(s)}\ ds \\ \widehat{c} &= \widehat{\sigma}(\alpha_2) - \widehat{\sigma}(\alpha_1) \\ \widehat{\sigma}(t) &= inf\{\sigma(s) + 2 \mid \int_s^t \sqrt{W_0(u)}\ du \mid, s \in \mathrm{R}\} \end{aligned}$$

c is the surface tension. Note that $| \widehat{c} | \leq c$ and that the ratio $\widehat{c}/c = \cos\theta$ gives the contact angle θ between the interface and the wall, following Young's law.

Let us now list some other dependances of W, σ, m and Ω with respect to ε, which lead to different models.

The form chosen by L. Modica for W ($W = W_0/\varepsilon$) ensure that c is a finite non vanishing quantity, but it avoids any compressibility effect (the values of u in the phases are prescribed). A way to include some compressibility effects is to consider a family of functions W_ε so that, as ε tends to zero, $\int_{\alpha_1}^{\alpha_2} \sqrt{W_\varepsilon(u)}\ du$ tends to infinity and $d^2W_\varepsilon/du^2(\alpha_1)$ remains finite. That was done by Buttazo and Al.[3] who considered the family $W_\varepsilon(u) = W_0(u) + 1/\varepsilon^3\ \psi(u/\varepsilon)$ where W_0 is of the type described in figure 1 and ψ is a positive function with $supp(\psi) \subset\subset\ [\alpha_1, \alpha_2]$.

When a liquid film lies on a wall, the vicinity of the interface and the wall leads to strengths which may stabilize or not the film [4]. The dependence of the total energy with respect to the thickness of the film can be obtained by letting m depend on ε in such a way that it converges to $\int_{\Omega} \alpha_2\ dx$.

In this paper we concentrate on the effects due to the rugosity of the wall already pointed out for their relationship with friction and hysteresis phenomenon [5] (i.e. the difference between the receeding and the advancing contact angle).

We will then assume that the boundary of Ω is oscillating with period d_ε and amplitude h_ε tending to zero with ε: Ω depends upon ε.

In the second section we will give a precise mathematical setting of the problem and state the main theorem of convergence (Γ-convergence of associated functionals). We are led to a relaxed boundary contact energy $\widehat{\widehat{\sigma}}$ which is related to a local problem.

In section 3 we study the influence of the surface parameters (*rugosity* parameters) upon this energy $\widehat{\widehat{\sigma}}$. We show that, for a given fluid and a given material for the wall, the rugosity parameters can increase the contact angle continuously from its value on a flat wall to π (complete wetting).

Section 4 is devoted to proofs.

figure 1

2.The main theorem

Let Ω be a bounded open subset $\mathrm{R^N}$ with a smooth boundary. An element $x \in \mathrm{R^N}$ is described by its coordinates in an orthogonal basis. As the last coordinate x_N plays a special role in our problem we shall write $x = (x', x_N)$ where $x' = (x_1, x_2, ..., x_{N-1})$.

The oscillating boundary is expressed in terms of a function $f : \mathrm{R^{N-1}} \longrightarrow [0,1]$ which is assumed to be C^1 and Y-periodic where $Y =]-1/2, 1/2[^{\mathrm{N-1}}$. Define for every positive d and h:

$$\Delta(d,h) = \{x_N > -hf(x'/d)\}$$
$$\Lambda(d,h) = \{x_N = -hf(x'/d)\}$$

Let d_ε and h_ε be two sequences of positive parameters tending to zero as $\varepsilon \longrightarrow 0$. We consider the following subsets of R^{N} (see figure 2):

$$\begin{array}{ccccccc}
\Omega_\varepsilon & = & \Delta(d_\varepsilon, h_\varepsilon) \cap \Omega & ; & \Omega_0 & = & \Delta(0,1) \cap \Omega \\
\Gamma_\varepsilon & = & \Lambda(d_\varepsilon, h_\varepsilon) \cap \Omega & ; & \Gamma_0 & = & \Lambda(0,1) \cap \Omega \\
\partial\Omega_\epsilon & = & \Gamma_\varepsilon \cup \Gamma' & ; & \partial\Omega_0 & = & \Gamma_0 \cup \Gamma'
\end{array}$$

figure 2

A zooming of a part of Γ_ε brings us to consider, λ and α being two real parameters, the subset: $B_\lambda = \lambda Y \times \mathrm{R}$ (see figure 3) and the following subsets of L^2_{loc}:

$$\mathcal{A}^L_\lambda(\alpha) = \{u \in H^1_{loc}(\Delta(1,1));\ u\ \lambda Y \text{ periodic in } x';\ u = \alpha \text{ for } x_N > L\}$$
$$\mathcal{B}^L_\lambda(\alpha) = \{u \in \mathcal{A}^L_\lambda(\alpha);\ u = \alpha \text{ on } \partial B_\lambda \cap \{x_N > 0\}\}$$

figure 3

The multiphase problem with the rough boundary Γ_ε reads as:

$$(\mathcal{P}^\varepsilon) \qquad inf\ \{E^\varepsilon(u),\ u \in L^2(\Omega),\ \int_\Omega u = m\} \qquad \text{where :}$$

$$E^\varepsilon(u) = \int_{\Omega_\varepsilon} [\varepsilon \mid Du \mid^2 + \frac{1}{\varepsilon} W(u)]dx + \int_{\partial\Omega_\varepsilon} \sigma(u)\ dH^{N-1}$$

if $u \in H^1(\Omega_\varepsilon)$ and $u = 0$ on $\Omega\backslash\overline{\Omega_\varepsilon}$,

$E^\varepsilon(u) = +\infty$ otherwise.

We will make the following assumptions:

$(H1)$ $\quad \lim \frac{h_\varepsilon}{d_\varepsilon} = \delta;\ \ \lim \frac{\varepsilon}{d_\varepsilon} = \gamma; \quad \delta, \gamma \in]0, +\infty[$

$(H2a)$ $\quad W$ is C^2; $W \geq 0$ and satisfies :

$(\ \ b)$ $\quad W(u) = 0 \Longleftrightarrow u \in \{\alpha_1, \alpha_2\}$

$(\ \ c)$ $\quad W - W^{**} \leq M$ for a suitable constant M

$(\ \ d)$ $\quad W(u) \geq \lambda_0 \mid u \mid^2 - \mu_0$ where $\lambda_0 > 0$

$(\ \ e)$ $\quad W''(\alpha_i) > 0$ (finite compressibility of the two phases)

$(H3)$ $\quad \sigma(u)$ is continuous; $\sigma(\mathrm{u}) \geq 0$ and $\sigma - \sigma^{**} \leq M$

Here W^{**} and σ^{**} denote the convexification of W and σ.

Our main result states that solutions of $\mathcal{P}^\varepsilon$ converge in $L^2(\Omega)$ to the solutions of a problem $\mathcal{P}^0$. To express this limit problem $\mathcal{P}^0$ we need the space $BV(\Omega_0)$ of functions $u \in L^1(\Omega_0)$ such that $\mid Du \mid (\Omega_0) = \sup\{\int_{\Omega_0} u\ div\ g\ dx\ ;\ g \in C_0^1(\Omega; \mathrm{R}^\mathrm{N}), |g| \leq 1\} < +\infty$.

We also use the following surface energies:

$$c = 2\int_{\alpha_1}^{\alpha_2} \sqrt{W(s)}\ ds \tag{2.1}$$

$$\widehat{c} = \widehat{\sigma}(\alpha_2) - \widehat{\sigma}(\alpha_1) \tag{2.2}$$

$$\widehat{\sigma}(\alpha) = inf\{\sigma(s) + 2 \mid \int_s^\alpha \sqrt{W(u)}\ du \mid, s \in \mathrm{R}\} \tag{2.3}$$

$$\widehat{\widehat{c}} = \widehat{\widehat{\sigma}}(\alpha_2) - \widehat{\widehat{\sigma}}(\alpha_1) \tag{2.4}$$

$$\widehat{\widehat{\sigma}}(\alpha) = \inf_{L>0} \inf_{u \in \mathcal{A}_1^L(\alpha)} \{ \int_{B\cap\Delta(1,1)} \{ \frac{\delta}{\gamma} W(u) + \frac{\gamma}{\delta} \mid \frac{\partial u}{\partial x_N} \mid^2 + \gamma\delta \mid \frac{\partial u}{\partial x'} \mid^2 \} \, dx + \\ + \int_{B\cap\Lambda(1,1)} a_\delta(x') \, \sigma(u) dH^{N-1} \} \tag{2.5}$$

where a_δ is a distortion factor defined as:

$$a_\delta = \frac{(1+\delta^2 \mid \nabla f \mid^2)^{1/2}}{(1+ \mid \nabla f \mid^2)^{1/2}} \tag{2.6}$$

Now $\mathcal{P}^0$ reads as:

$(\mathcal{P}^0)$ $\qquad \inf \{E^0(u), \ u \in L^2(\Omega), \int_\Omega u = m\} \qquad$ where :

$$E^0(u) = \frac{c}{\alpha_2 - \alpha_1} \int_{\Omega_0} \mid Du \mid \ dx + \int_{\Gamma'} \widehat{\sigma}(u) \ dH^{N-1} + \int_{\Gamma_0} \widehat{\widehat{\sigma}}(u) \ dH^{N-1}$$

if $u \in BV(\Omega)$, $u(x) \in \{\alpha_1, \alpha_2\}$ a.e. on Ω_0, $u(x) = 0$ a.e. on $\Omega \backslash \Omega_0$,

$E^0(u) = +\infty$ otherwise.

Theorem 2.1. *Under (H1), (H2), (H3), the sequence* E^ε Γ*-converges to* E^0 *in* $L^2(\Omega)$*, that is:*

(i) *For every sequence* (u_ε) *converging to* $u \in L^2(\Omega)$*, one has :* $\liminf_{\varepsilon \longrightarrow 0} E^\varepsilon(u_\varepsilon) \geq E^0(u)$

(ii) *For every* $u \in L^2(\Omega)$*, there exists a sequence* (u_ε) *such that :* $u_\varepsilon \longrightarrow u$ in $L^2(\Omega)$, $\limsup_{\varepsilon \longrightarrow 0} E^\varepsilon(u_\varepsilon) \leq E^0(u)$

Moreover we can choose u_ε *so that* $\int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx$ *holds for every* ε.

We notice that for $E^0(u) < +\infty$, u takes the form $u = \alpha_1 1_A + \alpha_2 1_{\Omega_0 \backslash A}$ where $A \subset \Omega_0$ is a measurable subset with a finite perimeter in Ω_0 that is: $\int_{\Omega_0} |D1_A| < +\infty$. Denoting by $\partial^* A$ the reduced boundary of A (see the book by Giusti [6] for all related concepts), we find that $(\mathcal{P}^0)$ reduces to a purely geometrical problem with respect to A (liquid drop problem):

$$(\widetilde{\mathcal{P}}^0) \qquad \inf \{c \, H^{N-1}(\partial^* A \cap \Omega_0) - \widehat{c} \, H^{N-1}(\partial^* A \cap \Gamma') + \\ - \widehat{\widehat{c}} \, H^{N-1}(\partial^* A \cap \Gamma_0) \ ; \ \ A \subset \Omega_0, \ \mid A \mid = m_1\}$$

where $m_1 = \dfrac{|\Omega| \ \alpha_2 - m}{\alpha_2 - \alpha_1}$, $m_1 \in [0, |\Omega|]$.

As a consequence of theorem 2.1 we get the convergence of $(\mathcal{P}^\varepsilon)$:

Theorem 2.2. *Fix* $m \in \,]\alpha_1|\Omega|, \alpha_2|\Omega|[$ *(existence of two phases). Let* (u_ε) *be a sequence in* $L^2(\Omega)$ *such that:*

$$E^\varepsilon(u_\varepsilon) - \inf\ \mathcal{P}^\varepsilon \ = o(\varepsilon) \quad , \quad \int_\Omega u_\varepsilon \ dx = m$$

Then (u_ε) *is relatively compact and every cluster point* u *is a solution of* $(\mathcal{P}^0)$ *that is* $u = \alpha_1 \ 1_A + \alpha_2 \ 1_{\Omega_0 \setminus A}$ *where* A *is solution of the liquid drop problem* $(\widetilde{\mathcal{P}}^0)$.

Comment: The wetting properties of the rough wall Γ_0 will be deduced from the ratio $\widehat{\widehat{c}}/c$ and compared with that of the flat wall Γ' characterized by $\widehat{c}/c$. These ratios will be shown to be in [-1,1] (see Sec. 3) and have a precise geometrical meaning: they are the cosine of the contact angle between the fluid phase α_1 and the wall in case this contact occurs on Γ_0 or on Γ'.

3. Estimates on the local problem and dependence of the homogenized boundary energy with respect to the rugosity parameters.

In what follows, α is assigned to take value α_1 or α_2. Given $\delta,\ \gamma \in]0,+\infty[$, it is convenient to define for every $u \in H^1_{loc}$ and B Borel subset of $\mathrm{R^N}$:

$$G(u,B) = \int_{\Delta(1,1)\cap B} \left(\frac{\delta}{\gamma}\ W(u) + \frac{\gamma}{\delta} \mid \frac{\partial u}{\partial x_N} \mid^2 + \delta\gamma \mid \frac{\partial u}{\partial x'} \mid^2\right) dx \\ + \int_{\Lambda(1,1)\cap B} a_\delta(x')\sigma(u^+)\ dH^{\mathrm{N}-1} \tag{3.1}$$

u^+ denoting the trace from above of u on $\Lambda(1,1)$.

We will use (also in sec. 4), the following local problems:

$$\sigma^L_\lambda(\alpha) = \inf\ \{\frac{G(u,B_\lambda)}{\lambda^{N-1}}\ ;\ u \in \mathcal{A}^L_\lambda(\alpha)\} \tag{3.2}$$

$$\tau^L_\lambda(\alpha) = \inf\ \{\frac{G(u,B_\lambda)}{\lambda^{N-1}}\ ;\ u \in \mathcal{B}^L_\lambda(\alpha)\} \tag{3.3}$$

where $\lambda,\ L$ are positive reals.

As $\widehat{\widehat{\sigma}}$ (see (2.5)) and the last expressions depend on δ and γ, we will sometimes write $\widehat{\widehat{\sigma}}(\gamma,\delta),\ \sigma^L_\lambda(\alpha,\gamma,\delta),\ \tau^L_\lambda(\alpha,\gamma,\delta)$ and $G_{\gamma,\delta}(u,B)$.

It is easy to check that $\sigma_\lambda^L(\alpha)$ and $\tau_\lambda^L(\alpha)$ are decreasing functions of L. Moreover:

$$\widehat{\widehat{\sigma}}(\alpha) = \lim_{L \longrightarrow \infty} \searrow \sigma_1^L(\alpha) \tag{3.4}$$

$$\sigma_\lambda^L(\alpha) \leq \tau_\lambda^L(\alpha) \qquad \forall\lambda,\ \forall L. \tag{3.5}$$

The first estimate (Proposition 3.1) will be crucial for the proof of theorem 2.1.

Proposition 3.1
i) For every $\delta,\ \gamma, L \in]0,+\infty[$ *and every* integer λ*, one has:*

$$\sigma_\lambda^L(\alpha) = \sigma_1^L(\alpha) \tag{3.6}$$

ii) Let I *a compact interval in* $]0,+\infty[$*. Then the following inequalities hold uniformly for* $\gamma, \delta \in I$ *and for suitable* $C,\ C'(L)$*:*

$$\sigma_1^L(\alpha) + \frac{C}{\lambda^{N-1}} \leq \tau_\lambda^L(\alpha) \leq \sigma_1^L(\alpha) + \frac{C'(L)}{\lambda^{(N-1)/2}} \tag{3.7}$$

$$\sigma_1^L(\alpha) \leq \widehat{\widehat{\sigma}}(\alpha) \leq \sigma_1^L(\alpha) + o(L) \tag{3.8}$$

Comments:

a) Let us stress the fact that the equality (3.6) is not trivial since the functional involved G is non-convex. In the homogenization theory, several non-convex examples exhibit a gap between the average energy on periodic cells of length $\lambda \in \mathrm{N}$ and the minimal energy taken on a unit cell (see S. Müller [7]). Fortunately this gap does not appear in our problem.

b) The first inequality in (3.7) is straightforward. The procedure to obtain the second one consists in taking the solution u^L associated with $\sigma_1^L(\alpha)$ and in using cutt-off functions on $B_\lambda \backslash B_{\lambda'}$ (where $\lambda' \in \mathrm{N}$) and $\lambda' \leq \lambda - 1$) in order to fit the boundary condition $u = \alpha$ on $\partial B_\lambda \cap \{x_N > 0\}$ (cf. the definition of $\mathcal{B}_\lambda^L(\alpha)$ in sec. 2). The parameter $C'(L)$ depends on the norm of $(u^L - \alpha)$ in $L^2(B_1)$.

To prove prosition 3.1 we need the following lemma:

Lemma 3.2 *The variational problem associated with* $\sigma_\lambda^L(\alpha)$ *has at least one solution. Moreover:*
i) If $\lambda = k$ *is an integer and* u *is a solution, so is* $u(x' - i, x_N)$ *for every* $i \in \{1,\ldots,k-1\}^{N-1}$
ii) If u *and* v *are two solutions, so are* $u \wedge v$ *and* $u \vee v$*.*

Proof. The existence of a solution is obtained classically by the weak compactness of minimizing sequences in the closed subset $\mathcal{A}_\lambda^L(\alpha)$ of H^1_{loc} and by the lowersemicontinuity of $G(.,B_\lambda)$ for this topology.

Assertion (i) is obvious since $u(x'-i,x_N)$ is still kY periodic in x' and agrees with α for $x_N > L$.

Let us prove that $w = u \wedge v$ and $w' = u \vee v$ are also solutions associated with $\sigma_\lambda^L(\alpha)$. Let:

$$A = \{x \in B_\lambda \cap \Delta(1,1)\ ;\ u(x) < v(x)\} \cup \{x \in B_\lambda \cap \Lambda(1,1)\ ;\ u^+(x) < v^+(x)\}$$

($u^+,\ v^+$ denote the traces of $u,\ v$ on $\Lambda(1,1)$ from above)
We have:

$$\begin{aligned} Dw &= 1_A\ Du + 1_{A^c}\ Dv \quad \text{a.e. on } B_\lambda \cap \Delta(1,1) \\ w^+ &= 1_A\ u^+ + 1_{A^c}\ v^+ \quad H^{N-1} \text{ a.e. on } B_\lambda \cap \Lambda(1,1) \end{aligned}$$

Hence:

$$\begin{aligned} G(w, B\lambda) &= G(u, B_\lambda \cap E) + G(v, B_\lambda \cap A^c) \\ G(w', B\lambda) &= G(v, B_\lambda \cap E) + G(u, B_\lambda \cap A^c) \end{aligned}$$

By adding these two equalities, one gets:

$$G(w, B_\lambda) + G(w', B_\lambda) = G(u, B_\lambda) + G(v, B_\lambda) = 2\ \sigma_\lambda^L(\alpha)$$

Since w and w' are also in the admissible set $\mathcal{A}_\lambda^L(\alpha)$, one has:

$$G(w, B_\lambda) = G(w', B_\lambda) = \sigma_\lambda^L(\alpha)$$

□

Proof of Proposition 3.1

$$\text{Define} : \widehat{u}(x', x_N) = \inf_{i \in \{1,2,\ldots,k-1\}^{N-1}} u(x' - i, x_N)$$

From lemma 3.2 we deduce that $\widehat{u}$ is a solution associated with $\sigma_\lambda^L(\alpha)$ ($\lambda \in \mathrm{N}$)

Obviously $\widehat{u}$ satisfies $\widehat{u} = \alpha$ for $x_N > L$ and $\widehat{u}(., x_N)$ is Y-periodic. Hence $\widehat{u} \in \mathcal{A}_1^L(\alpha)$ and:

$$\sigma_\lambda^L(\alpha) = \frac{G(\widehat{u}, B_\lambda)}{\lambda^{N-1}} = G(\widehat{u}, B_1) = \sigma_1^L(\alpha)$$

Let us just outline the proof of (ii). Let $\delta_L,\ \gamma_L$ in I. As I is compact, we can assume that, as $L \longrightarrow \infty$, $\delta_L \longrightarrow \delta,\ \gamma_L \longrightarrow \gamma$ and that u_L, a solution associated with $\sigma_1^L(\alpha,\ \gamma_L,\ \delta_L)$ converges weakly to some u in H^1_{loc}. Using the weak lower semicontinuity in H^1_{loc} of $G(., B_1)$, one gets:

$$\widehat{\widehat{\sigma}}(\alpha) = \lim_{L \longrightarrow +\infty} \sigma_1^L(\alpha) = \lim_{L \longrightarrow +\infty} G(u_L, B_1) \geq G(u, B_1) \tag{3.9}$$

Let us define:

$$\psi(s) = 2\int_\alpha^s \sqrt{W(t)}dt\ ,\ v_L = \psi(u_L)\ ,\ v = \psi(u)$$

Since, for suitable $C_0 > 0$:

$$G(u_L, B_1) \geq C_0 \int_{B_1 \cap \Delta(1,1)} |\ Dv_L\ |\ \ dx$$

we obtain:

$$\int_{B_1 \cap \Delta(1,1)} | \, Dv \, | \;\; dx \leq \liminf_{L \longrightarrow +\infty} \int_{B_1 \cap \Delta(1,1)} | \, Dv_L \, | \;\; dx < +\infty$$

From assumptions (H2c), (H2d) and (H2e), we can show that $\psi(s) \geq C_1 \;\; | \, s - \alpha \, |^2$ for some suitable $C_1 > 0$. Then:

$$\int_Y | \, u - \alpha \, |^2 \; (x', t) \; dx' \leq \frac{1}{C_1} \int_{Y \times \{x_N > t\}} | \, Dv \, |$$

Integrating this inequality between $L - 1$ and L gives:

$$\beta_L = \int_{Y \times [L-1, L]} | \, u - \alpha \, |^2 \;\; dx \;\; \longrightarrow 0 \;\; \text{as } L \longrightarrow +\infty$$

We conclude by multiplying u by a suitable cutt-off function $\varphi(x_N)$ on $[L-1, L]$ (see the proof of lemma 4.2 below for a similar construction). That yields:

$$\sigma_1^L(\alpha) \leq G(u, B_1) + o(\beta_L)$$

Combined with (3.9), we obtain (3.7). □

Proposition 3.3

i) $\widehat{\widehat{\sigma}}(\alpha, \gamma, \delta)$ *is continuous with respect to* $\delta, \;\gamma \in]0, +\infty[$.

ii) For every $\gamma, \;\delta \in]0, +\infty[$, *we have:*

$$|\widehat{\widehat{c}}(\gamma, \delta)| = |\widehat{\widehat{\sigma}}(\alpha_2, \gamma, \delta) - \widehat{\widehat{\sigma}}(\alpha_1, \gamma, \delta)| \leq c$$

Proof. The continuity of $\sigma_1^L(\alpha, ., .)$ for every $L > 0$ is straightforward. Then (i) is deduced by using estimate (3.8). For (ii) we prove the inequality $\widehat{\widehat{\sigma}}(\alpha_i) \leq \widehat{\widehat{\sigma}}(\alpha_j) + c$ for $i \neq j$ by extending a solution associated with the definition of $\sigma_1^L(\alpha_j)$ (L being fixed) by a function φ depending only on x_N such that $\varphi(L) = \alpha_j$ and $\varphi(+\infty) = \alpha_2$. □

Now we have to pay attention to the ratio $\widehat{\widehat{c}}/\widehat{c}$ which determines the contact angle on the rough surface associated with γ and δ. This ratio is ruled by the propositions 3.4 and 3.5 below, for which we leave out the proofs in order to be concise.

Proposition 3.4

i) Let us fix $\gamma \in]0, +\infty[$. Then:

$$\lim_{\delta \longrightarrow 0^+} \widehat{\widehat{\sigma}}(\alpha, \gamma, \delta) = \widehat{\sigma}(\alpha) \ , \quad \lim_{\delta \longrightarrow 0^+} \widehat{\widehat{c}}(\gamma, \delta) = \widehat{c}$$

ii) Assume that σ does not reach a minimum on $]\alpha_1, \alpha_2[$. Then:

$$\lim_{\delta \longrightarrow +\infty} \widehat{\widehat{c}}(\gamma, \delta) = c$$

Proof. Assertion (i) is obtained by approximating the infimum associated with $\sigma_1^L(\alpha, \gamma, \delta)$ by a function depending only on x_N and constant for $x_N < 0$. □

Comment: By the continuity of $\widehat{\widehat{c}}(\gamma, .)$, we see that $\widehat{\widehat{c}}(\gamma, .)$ ranges onto the interval $[c, \widehat{c}]$. In other words, every situation between the case of a flat surface ($\widehat{\widehat{c}} = \widehat{c}$) and the perfectly wetting case ($\widehat{\widehat{c}} = c$) is reached by increasing the slope factor δ from 0 to $+\infty$.

When the scale of rugosity is large with respect to the thickness of the phase transition ($\gamma << 1$), we are led to a local anisotropic Plateau problem:

Recalling that a_δ is given by (2.6), we define, for every fixed $\delta \in]0, +\infty[$, the quantity R_δ and the positively 1-homogeneous convex function $h_\delta(p)$ on $\mathrm{R^N}$ by:

$$h_\delta(p) = ({p_N}^2 + \delta^2 \ p'^2)^{1/2} \ , \quad R_\delta = \int_Y (1 + \delta^2 |\nabla f|^2)^{1/2} \ dx'$$

Proposition 3.5:

$$i) \quad \lim_{\gamma \longrightarrow 0^+} \widehat{\widehat{\sigma}}(\alpha, \gamma, \delta) = \inf_{u \in \mathcal{C}(\alpha)} \{ \frac{c}{\alpha_2 - \alpha_1} \int_{B_1 \cap \Delta(1,1)} h_\delta(Du) \ dx \ + \\ + \int_{B_1 \cap \Lambda(1,1)} a_\delta \ \widehat{\sigma}(u) dH^{\mathrm{N}-1} \}$$

where

$$\mathcal{C}(\alpha) = \{ u \in BV_{loc}(\Delta(1,1)); \ u(x) \in \{\alpha_1, \alpha_2\} \text{ a.e.}; \ u = \alpha \text{ for } x_N > 0 \}$$

ii) $\widehat{\widehat{c}}(+\infty, \delta)$ *is continuously increasing with respect to δ. Moreover:*

$$\widehat{\widehat{c}}(+\infty, \delta) = R_\delta \ \widehat{c} \quad \text{for } o \leq \delta \leq \frac{[(\frac{c}{\widehat{c}})^2 - 1]^{1/2}}{Lip(f)}$$

$$\lim_{\delta \longrightarrow +\infty} \widehat{\widehat{c}}(+\infty, \delta) = c$$

Proof. For assertion (i), we refer to Bouchitte [9] where limits of phase transition models with general anisotropic perturbations are described. □

4. Proofs of theorems 2.1 and 2.2

It will be convenient to localize the energy associated with E^ε as follows; let us define for every $u \in L^2(\Omega)$ and every Borel subset $B \subset \overline{\Omega}$:

$$F_{\varepsilon,d,h}(u,B) = \int_{B\cap\Delta(d,h)} [\varepsilon|Du|^2 + \frac{1}{\varepsilon}W(u)]\ dx + \int_{B\cap(\Lambda(d,h)\cup\Gamma')} \sigma(u^+)\ dH^{\mathrm{N}-1}$$

if $u \in H^1(B \cap \Delta(d,h))$,

$$F_{\varepsilon,d,h}(u,B) = +\infty \qquad \text{otherwise.}$$

For simplicity $F_{\varepsilon,d_\varepsilon,h_\varepsilon}$ will be denoted F_ε so that, if $u = 0$ on $\Omega\backslash\Omega_\varepsilon$, one has $E^\varepsilon(u) = F_\varepsilon(u,\overline{\Omega\varepsilon})$.

We will use the following lemmas:

Lemma 4.1 (scaling) *Let $\varepsilon,\ h,\ d\ > 0$, real parameters. Then, for every $\lambda > 0$ and every v in $L^1(Q)$, one has:*

$$F_{\lambda\varepsilon,\lambda h,\lambda d}(v,Q_\lambda) = \lambda^{N-1}\ F_{\varepsilon,h,d}(v_\lambda,Q)$$

where $v_\lambda(x) = v(\lambda x)$.

Proof. We use the change of variable $y = x/\lambda$ so that $\nabla v(x) = \lambda^{-1}\ \nabla v_\lambda(y)$. □

Lemma 4.2 *Let $u_0 = \alpha$ for $x_N > 0$ and $u_0 = 0$ for $x_N < 0$ with $\alpha \in \{\alpha_1,\alpha_2\}$ and assume that h/ε remains bounded. Then for every sequence $u_\varepsilon \in L^1(Q)$ such that $u_\varepsilon \longrightarrow u_0$ there exists $v_\varepsilon \in L^1(Q)$ such that:*

(i) $\qquad v_\varepsilon \in H^1(Q\cap\{x_N > 0\})\ ;\quad v_\varepsilon = \alpha$ on $\partial Q \cap \{x_N > 0\}$

(ii) $\qquad \liminf_{\varepsilon\longrightarrow 0}\ F^\varepsilon(v_\varepsilon,Q) \le \liminf_{\varepsilon\longrightarrow 0}\ F^\varepsilon(u_\varepsilon,Q)$

(iii) $\qquad v_\varepsilon \longrightarrow u_0$ in $L^1(Q)$

Proof. First, possibly by extracting a subsequence, we can assume:

$$\liminf_{\varepsilon\longrightarrow 0}\ F^\varepsilon(v_\varepsilon,Q) = \lim_{\varepsilon\longrightarrow 0}\ F^\varepsilon(u_\varepsilon,Q) = \beta < +\infty \tag{4.1}$$

Hence

$$\lim_{\varepsilon\longrightarrow 0}\int_Q W(u_\varepsilon)\ dx = 0$$

and from growth condition (H2c), the sequence $\mid u_\varepsilon \mid^2$ is uniformly integrable. Thus:

$$u_\varepsilon \longrightarrow u_0 \text{ in } L^2(Q) \tag{4.2}$$

Let $h_\varepsilon,\ k_\varepsilon$ be sequences such that:

$$k_\varepsilon \in \mathrm{N},\quad \varepsilon k_\varepsilon \longrightarrow 1 \quad (\text{hence } k_\varepsilon \longrightarrow\ +\infty) \tag{4.3}$$

$$h_\varepsilon = o(|u_\varepsilon - u_0|)_{L^2(Q)} \tag{4.4}$$

Then we define a boundary layer on Q by setting:

$$T_\varepsilon = Q \backslash (1-h_\varepsilon)Q \tag{4.5}$$

We divide T_ε into k_ε slices of width $h_\varepsilon/k_\varepsilon$.

$$S^i_\varepsilon = Q^i_\varepsilon \backslash Q^{i-1}_\varepsilon,\quad Q^i_\varepsilon = (1-i\ h_\varepsilon/k_\varepsilon)Q, \quad i \in \{0,1,...,k_\varepsilon\} \tag{4.6}$$

so that: $T_\varepsilon = \cup_{i=1}^{k_\varepsilon}\ S^i_\varepsilon$.

Let φ^i_ε be a smooth function such that:

$$0 \le \varphi^i_\varepsilon \le 1, \quad \varphi^i_\varepsilon = 1 \text{ on } Q^i_\varepsilon,\ \ \varphi^i_\varepsilon = 0 \text{ on } Q\backslash Q^{i-1}_\varepsilon, \mid D\varphi^i_\varepsilon \mid \le \frac{h_\varepsilon}{k_\varepsilon} \tag{4.7}$$

For a suitable i (we shall choose later), let us consider $v_\varepsilon = u_\varepsilon \varphi^i_\varepsilon + \alpha\ (1-\varphi^i_\varepsilon)$. We have $v_\varepsilon = \alpha$ on $\partial Q \cap \{x_N > 0\}$, and owing to assumptions (H2c) and (H3), the following inequalities hold:

$$W(v_\varepsilon) \le W(u_\varepsilon) + C, \quad\ (W(\alpha)=0) \tag{4.8}$$

$$\sigma(v_\varepsilon) \le \sigma(u_\varepsilon) + C, \quad\ \text{on } \Gamma_\varepsilon \cap Q \tag{4.9}$$

On the other hand from (4.7) $v_\varepsilon = u_\varepsilon$ on $Q^i_\varepsilon, v_\varepsilon = \alpha$ on $Q\backslash Q^{i-1}_\varepsilon$ and $Dv_\varepsilon = \varphi^i_\varepsilon Du_\varepsilon + (u_\varepsilon - \alpha)\ D\varphi^i_\varepsilon$, so we have:

$$\mid Dv_\varepsilon \mid^2 \le 2 \mid Du_\varepsilon \mid^2 + \mid u_\varepsilon - \alpha \mid^2\ \ (k_\varepsilon/h_\varepsilon)^2 \quad\ \text{on } S^i_\varepsilon \tag{4.10}$$

$$\begin{aligned} F^\varepsilon(v_\varepsilon, Q) &= F^\varepsilon(u_\varepsilon, Q^i_\varepsilon) + F^\varepsilon(v_\varepsilon, S^i_\varepsilon) + F^\varepsilon(\alpha, Q\backslash Q^{i-1}_\varepsilon) \\ &\le F^\varepsilon(u_\varepsilon, Q) + R^i_\varepsilon \end{aligned} \tag{4.11}$$

where

$$R^i_\varepsilon = 2\ \varepsilon \int_{S^i_\varepsilon \cap \Delta_\varepsilon} \mid Du_\varepsilon \mid^2 + \frac{2\ \varepsilon k^2_\varepsilon}{h^2_\varepsilon} \int_{S^i_\varepsilon \cap \Delta_\varepsilon} \mid u_\varepsilon - \alpha \mid^2\ + \frac{c}{\varepsilon} \mid S^i_\varepsilon \mid + c\ H^{N-1}(S^i_\varepsilon \cap \Gamma_\varepsilon) \tag{4.12}$$

Choose i such that $R^i_\varepsilon \le \sum_{j=1}^{k_\varepsilon} R^j_\varepsilon / k_\varepsilon$. One gets from (4.12):

$$\begin{aligned} R^i_\varepsilon \le \frac{2\ \varepsilon}{k_\varepsilon} \int_{T_\varepsilon \cap \Delta_\varepsilon} \mid Du_\varepsilon \mid^2 + \frac{2\ \varepsilon k^2_\varepsilon}{h^2_\varepsilon} \int_{T_\varepsilon \cap \Delta_\varepsilon} \mid u_\varepsilon - u_0 \mid^2 \\ +\ \frac{c\ h^{N-1}_\varepsilon}{\varepsilon k_\varepsilon} + \frac{c}{k_\varepsilon} H^{N-1}(T_\varepsilon \cap \Gamma_\varepsilon) \end{aligned}$$

Noticing that: $\varepsilon \int_{T_\varepsilon \cap \Delta_\varepsilon} \mid Du_\varepsilon \mid^2 \leq F^\varepsilon(u_\varepsilon, Q)$ and $H^{N-1}(T_\varepsilon \cap \Gamma_\varepsilon) \leq H^{N-1}(Q \cap \Gamma_\varepsilon) \leq \sup\ (1 + h_\varepsilon/d_\varepsilon \mid Df \mid^2)^{1/2}$ we obtain owing to (4.1),(4.3),(4.4) and to the assumption that $h_\varepsilon/d_\varepsilon$ is bounded:

$$\limsup_{\varepsilon \longrightarrow 0}\ R^i_\varepsilon = \lim_{\varepsilon \longrightarrow 0}\ R^i_\varepsilon = O$$

which yields by (4.11) to the inequality (ii). The assertion (iii) is trivial since $\mid v_\varepsilon - u_0 \mid \leq \mid u_\varepsilon - u_0 \mid$. □

Lemma 4.3 *Let* $Q =]-1/2, 1/2[^{N-1}$, $\alpha \in \{\alpha_1,\ \alpha_2\}$ *and* $(\varepsilon,\ d_\varepsilon,\ h_\varepsilon)$ *a sequence such that (H1) holds. Define* $u_0(x) = \alpha$ *if* $x_N > 0$, $u_0(x) = 0$ *if* $x_N < 0$. *Then:*

i) For every sequence (u_ε) *such that* $u_\varepsilon \longrightarrow u_0$ *in* $L^2(Q)$, *one has:*

$$\liminf_{\varepsilon \longrightarrow 0}\ F_\varepsilon(u_\varepsilon, Q) \geq \widehat{\widehat{\sigma}}(\alpha, \gamma, \delta)$$

ii) There exists (u_ε) *such that* $u_\varepsilon = \alpha$ *on* $\partial Q \cap \{x_N > 0\}$ *and:*

$$u_\varepsilon \longrightarrow u_0 \ \text{ in } L^2(Q)\ ,\ \ \limsup_{\varepsilon \longrightarrow 0}\ F_\varepsilon(u_\varepsilon, Q) \leq \widehat{\widehat{\sigma}}(\alpha, \gamma, \delta)$$

Proof. By lemma 4.2, one can assume that $u_\varepsilon = \alpha$ on $\partial Q \cap \{x_N > 0\}$. Apply the lemma 4.1 with $\lambda = 1/d_\varepsilon$; one gets:

$$F_{\varepsilon, d_\varepsilon, h_\varepsilon}(u_\varepsilon, Q) = {d_\varepsilon}^{N-1}\ F_{\varepsilon/d_\varepsilon, 1, h_\varepsilon/d_\varepsilon}(v_\varepsilon, Q_{1/d_\varepsilon})$$

where $v_\varepsilon(x) = u_\varepsilon(x/d_\varepsilon)$.

Noticing that v_ε belongs to $\mathcal{B}^{L_\varepsilon}_{L_\varepsilon}(\alpha)$ with $L_\varepsilon = 1/d_\varepsilon$, we deduce from proposition 3.1 (cf. (3.7)):

$$\begin{aligned} F_{\varepsilon, d_\varepsilon, h_\varepsilon}(u_\varepsilon, Q) &\geq \tau^{L_\varepsilon}_{L_\varepsilon}(\alpha, \varepsilon/d_\varepsilon, h_\varepsilon/d_\varepsilon) \geq \\ &\geq \widehat{\widehat{\sigma}}(\alpha, \varepsilon/d_\varepsilon, h_\varepsilon/d_\varepsilon) - \frac{C}{{L_\varepsilon}^{N-1}} \end{aligned}$$

The conclusion (i) follows by letting ε tend to 0 and using the continuity of $\widehat{\widehat{\sigma}}(\alpha, ., .)$ at (γ, δ) proved in proposition 3.2.

Now, let us prove assertion (ii): Let $w \in \mathcal{B}^1_1(\alpha)$ be the solution associated with the definition of $\tau^1_1(\alpha, \gamma, \delta)$. Define u_ε by:

$$\begin{aligned} u_\varepsilon(x) &= w(x'/d_\varepsilon, x_N/h_\varepsilon) \quad \text{if } x \in \Delta(d_\varepsilon, h_\varepsilon) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Through the $d_\varepsilon Y$-periodicity of u_ε with respect to x', we discover:

$$\int_{|x_N| \leq h_\varepsilon} |u_\varepsilon|^2\ dx = o(d_\varepsilon)$$

Since $u_\varepsilon = u_0$ for $|u_\varepsilon| > h\varepsilon$, one gets $u_\varepsilon \longrightarrow u_0$ in $L^2(Q)$. Now using the scaling lemma 4.1 with $\lambda = 1/d_\varepsilon$:

$$F_{\varepsilon,d_\varepsilon,h_\varepsilon}(u_\varepsilon, Q) = {d_\varepsilon}^{N-1}\ F_{\varepsilon/d_\varepsilon,1,h_\varepsilon/d_\varepsilon}(v_\varepsilon, Q_{1/d_\varepsilon})$$

where

$$v_\varepsilon(x) = w(x', \frac{d_\varepsilon}{h_\varepsilon} x_N)$$

Since $w = \alpha$ for $x_N > h_\varepsilon/d_\varepsilon$, we have $v_\varepsilon \in \mathcal{B}_1^L(\alpha)$ for every $L > \sup(h_\varepsilon/d_\varepsilon)$. According to proposition 3.1, we have:

$$\begin{aligned} F_{\varepsilon,d_\varepsilon,h_\varepsilon}(u_\varepsilon, Q) &\le \tau^L_{1/d_\varepsilon}(\alpha, \gamma_\varepsilon, \delta_\varepsilon) \\ &\le \sigma_1^L(\alpha, \gamma_\varepsilon, \delta_\varepsilon) + C'(L)\ {d_\varepsilon}^{\frac{N-1}{2}} \end{aligned}$$

Finally from the continuity of $\sigma_1^L(\alpha, ., .)$:

$$\limsup_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon) \le \sigma_1^L(\alpha, \gamma, \delta)$$

which reduces to the inequality of (ii) when $L \longrightarrow \infty$. □

4.1. Lowerbound for the Γ-limit of the energy

We are going to prove the assertion (i) of theorem 2.1.

Let (u_ε) be a sequence in $L^2(\Omega)$ such that:

$$u_\varepsilon \longrightarrow u \ \text{ in } L^2(\Omega)\ , \quad l = \liminf_{\varepsilon \longrightarrow 0}\ E^\varepsilon(u_\varepsilon) < +\infty \tag{4.13}$$

Define the Borel non-negative measure μ_ε on $\overline{\Omega}$ by setting:

$$\mu_\varepsilon(A) = F_\varepsilon(u_\varepsilon, A)$$

From (4.13), the sequence (μ_ε) is bounded and tight on $\overline{\Omega}$ ($\overline{\Omega}$ is compact). We can write, possibly only for a subsequence still denoted by ε:

$$\liminf_{\varepsilon \longrightarrow 0}\ E^\varepsilon(u_\varepsilon) = \lim_{\varepsilon \longrightarrow 0}\ E^\varepsilon(u_\varepsilon) = l$$

$$\mu_\varepsilon \longrightarrow \mu_0 \ \text{ for the narrow convergence on } \overline{\Omega}$$

As *supp* $\mu_\varepsilon \subset \overline{\Omega_\varepsilon}$, we have *supp* $\mu_0 \subset \overline{\Omega_0}$, hence:

$$l = \lim_{\varepsilon \longrightarrow 0}\ \mu_\varepsilon(\overline{\Omega}) = \mu_0(\overline{\Omega_0}) \tag{4.14}$$

In fact, by using the narrow convergence of μ_ε, for every Borel subset A of $\overline{\Omega}$ such that $\mu_0(\partial A) = 0$, one has:

$$\lim_{\varepsilon \longrightarrow 0}\ F_\varepsilon(u_\varepsilon, A) = \lim_{\varepsilon \longrightarrow 0}\ \mu_\varepsilon(A) = \mu_0(A) \tag{4.15}$$

Since $u_\varepsilon = 0$ on $\Omega\backslash\Omega_\varepsilon$ and:

$$E^\varepsilon(u_\varepsilon) \geq \varepsilon \int_{\Omega_0} |Du_\varepsilon|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega_0} W(u_\varepsilon) \, dx \ ,$$

we already know that the limit u lies in $BV(\Omega)$ and has the form: $u = \alpha_1 \, 1_{A\cap\Omega_0} + \alpha_2 \, 1_{\Omega_0\backslash A}$ where A is a subset of Ω_0 with finite perimeter. Moreover, by applying Modica's results [2], on the open set $\Omega^\eta = \Omega \cap \{x_N > \eta\}$, with $\eta > 0$, one gets:

$$\liminf_{\varepsilon\longrightarrow 0} \ F_\varepsilon(u_\varepsilon, \Omega^\eta) \geq \frac{c}{\alpha_2 - \alpha_1} \int_{\Omega^\eta} |Du| \, dx + + \int_{\Gamma'\cap\Omega^\eta} \widehat{\sigma}(u^+) \, dH^{\mathrm{N}-1} \tag{4.16}$$

where c and $\widehat{\sigma}$ are given by (2.1) and (2.3). Choosing a sequence η tending to 0 such that $\mu_0(\{x_N = \eta\}) = 0$ and using (4.15) and (4.16) we are led to:

$$\begin{aligned} l = \mu_0(\overline{\Omega_0}) &= \mu_0(\Gamma_0) + \lim_{\eta\longrightarrow 0} \mu_0(\Omega^\eta) \\ &\geq \mu_0(\Gamma_0) + \frac{c}{\alpha_2 - \alpha_1} \int_{\Omega_0} |Du| \, dx + \int_{\Gamma'} \widehat{\sigma}(u^+) \, dH^{\mathrm{N}-1} \end{aligned}$$

So the lowerbound of theorem 2.1 is obtained provided we can prove:

$$\mu_0 \, 1_{\Gamma_0} \geq \widehat{\widehat{\sigma}}(u^+) \ H^{\mathrm{N}-1}(\Gamma_0 \cap .)$$

Denoting $\theta_0 = H^{\mathrm{N}-1}(\Gamma_0 \cap .)$, the last inequality reads as:

$$\frac{d\mu_0}{d\theta_0}(x) \geq \widehat{\widehat{\sigma}}(u^+(x)) \ , \quad \theta_0 \text{ a.e. } x \in \Gamma_0 \tag{4.17}$$

Let Q_δ be the interval $]-\delta/2, +\delta/2[^N$. By the Besicovitch differentiation theorem, one has:

$$H^{\mathrm{N}-1} \text{ a.e. } x \in \Gamma_0 \ , \quad \frac{d\mu_0}{d\theta_0} = \lim_{\delta\longrightarrow 0} \frac{\mu_0(Q_\delta)}{\delta^{N-1}} \tag{4.18}$$

On the other hand, as the limit u of u_ε satisfies $u(x) \in \{\alpha_1, \ \alpha_2\}$ a.e. on $\Omega\backslash\Omega_0$, the traces on Γ_0 from $x_N > 0$ and from $x_N < 0$ satisfy :

$$u^+ \in \{\{\alpha_1, \ \alpha_2\} \quad H^{\mathrm{N}-1} \text{ a.e. } x \in \Gamma_0 \ , \quad u^- = 0$$

Let us fix $x_0 \in \Gamma_0$ such that the equality in (4.18) holds and let us set:

$$\begin{aligned} w_{\varepsilon,\delta}(y) &= u_\varepsilon(x_0 + \delta y) \\ w_{0,\delta}(y) &= u(x_0 + \delta y) \end{aligned}$$

It is well known that for $H^{\mathrm{N}-1}$ a.e. $x_0 \in \Gamma_0$, one has:

$$w_{0,\delta} \longrightarrow u_0 \ \text{ in } L^1(Q) \ \text{ as } \delta \longrightarrow 0 \tag{4.19}$$

where:

$$\begin{aligned} &u_0(y) = \alpha_i \ \text{ if } y_N > 0 \ \text{ and } u^+(x_0) = \alpha_i \\ &u_0(y) = 0 \quad \text{if } y_N < 0 \end{aligned} \tag{4.20}$$

From now on we assume that x_0 has been chosen in such a way that (4.18) and (4.19) hold.

Let us go on with the blow-up argument at x_0 as done in another situation by Fonseca-Müller [8]. Take a sequence (δ_k) tending to 0 such that $\mu_0(\partial Q_{\delta_k}) = 0$. Owing to (4.15) one has:

$$\frac{d\mu_0}{d\theta_0}(x_0) = \lim_{k \longrightarrow +\infty} \lim_{\varepsilon \longrightarrow 0} \frac{F_{\varepsilon,d_\varepsilon,h_\varepsilon}(u_\varepsilon, x_0 + Q_{\delta_k})}{\delta_k^{N-1}} \tag{4.21}$$

Assuming that $\frac{d\mu_0}{d\theta_0}(x_0) < +\infty$ (otherwise the inequality (4.17) is trivial), we can choose for every k, some $\varepsilon(k) > 0$ such that:

$$\varepsilon_k = \frac{\varepsilon}{\delta_k} \leq \frac{1}{k} \tag{4.22}$$

$$\| w_{\varepsilon,\delta_k} - u_0 \|_{L^1(Q)} \leq \| w_{0,\delta_k} - u_0 \|_{L^1(Q)} + \frac{1}{k} \tag{4.23}$$

$$\frac{F_{\varepsilon,d_\varepsilon,h_\varepsilon}(u_\varepsilon, Q_{\delta_k})}{\delta_k^{N-1}} \leq \frac{d\mu_0}{d\theta_0}(x_0) + \frac{1}{k} \tag{4.24}$$

For this ε (depending on k), let us set:

$$v_k(y) = u_\varepsilon(x_0 + \delta_k y)$$
$$d_k = \frac{d_\varepsilon}{\delta_k} \ , \quad h_k = \frac{h_\varepsilon}{\delta_k}$$

By lemma 4.1 we may rewrite (4.22) as:

$$F_{\varepsilon_k,d_k,h_k}(v_k, Q) \leq \frac{d\mu_0}{d\theta_0}(x_0) + \frac{1}{k}$$

We notice that $d_k \longrightarrow 0$, $h_k \longrightarrow 0$, $\varepsilon_k \longrightarrow 0$, while $h_k/d_k \longrightarrow \delta$, $\varepsilon_k/d_k \longrightarrow \gamma$. Since $v_k \longrightarrow u_0$ in $L^1(Q)$ where u_0 has the particular form given by (4.20), we may apply lemma 4.3 (i) with $\Omega = Q$. Combined with (4.24) that yields (4.17):

$$\frac{d\mu_0}{d\theta_0}(x_0) \geq \liminf_{k \longrightarrow +\infty} \ F_{\alpha_k,d_k,h_k}(v_k, Q) \geq \widehat{\widehat{\sigma}}(u^+(x_0))$$

□

4.2 Upperbound for the Γ-limit

We prove the assertion (ii) of theorem 2.1 in case $u = \alpha_1 \ 1_{\Omega_0 \cap A} + \alpha_2 \ 1_{\Omega_0 \setminus A}$ where A is an open set of $\mathrm{R^N}$ with smooth boundary ∂A such that $H^{\mathrm{N}-1}(\partial A \cap \partial \Omega_0) = 0$. The conclusion in the general case is then deduced by an approximation procedure for functions u such that $E^0(u) < +\infty$. This procedure is completely described in Modica [2] or Bouchitté [9] to which we refer for this part of the proof.

Set $A_1 = A \cap \Omega$ and $A_2 = \Omega \backslash A$. For every $\eta > 0$ let us define:

$$\Gamma^i_\eta = \{x \in \Gamma_0 \cap A_i \ ; \ \ d(x, \partial\Omega) > \eta, \ d(x, \partial A) > \eta\}$$
$$\Sigma^i_\eta = \Gamma^i_\eta \times [-\eta/2, \eta/2]$$

Noticing that

$$H^{\mathrm{N}-1}(\Gamma_\varepsilon \backslash (\Sigma^1_\eta \cup \Sigma^2_\eta)) \leq \frac{h_\varepsilon}{d_\varepsilon} \ H^{\mathrm{N}-1}(\Gamma_0 \backslash (\Gamma^1_\eta \cup \Gamma^2_\eta)),$$
$$\lim_{\eta \longrightarrow 0} H^{\mathrm{N}-1}(\Gamma_0 \backslash (\Gamma^1_\eta \cup \Gamma^2_\eta)) = H^{\mathrm{N}-1}(\Gamma_0 \cap (\Gamma' \cup \partial A)) = 0,$$

we can write:

$$\limsup_{\varepsilon \longrightarrow 0} H^{\mathrm{N}-1}(\Gamma_\varepsilon \backslash (\Sigma^1_\eta \cup \Sigma^2_\eta)) = o(\eta) \tag{4.25}$$

figure 4

Let us apply Modica's construction [2] to approach $\tilde{u} = \alpha_1 \ 1_{A_1} + \alpha_2 \ 1_{A_2}$ on the open subset $\Omega^{-\eta} = \Omega \cap \{x_N > -\eta\}$ taking into account the boundary energy on Γ'. We find a sequence $\tilde{u}_\varepsilon$ in $L^2(\Omega^{-\eta})$ such that:

$$\tilde{u}_\varepsilon \longrightarrow \tilde{u} \quad \text{in } L^2(\Omega^{-\eta}) \tag{4.26}$$

$$\tilde{u}_\varepsilon \quad \text{is bounded in} \ \ L^\infty(\Omega^{-\eta}) \tag{4.27}$$

$$\tilde{u}_\varepsilon = \tilde{u} \quad \text{if} \ \ d(x, \partial A) > \eta \quad \text{and} \ \ d(x, \partial\Omega^{-\eta}) > \eta \tag{4.28}$$

$$F_\varepsilon(\tilde{u}_\varepsilon, \Omega^{-\eta} \cup \Gamma' \backslash \Gamma_\varepsilon) \longrightarrow l_\eta \ \ \text{as} \ \ \varepsilon \longrightarrow 0 \quad \text{where}: \tag{4.29}$$

$$l_\eta = \frac{c}{\alpha_2 - \alpha_1} \int_{\Omega^{-\eta}} |D\tilde{u}| + \int_{\Gamma'} \widehat{\sigma}(u) \ dH^{\mathrm{N}-1} \tag{4.30}$$

(To obtain (4.27), we notice that, according to the growth condition on W (H2d), the infimum associated with the definition (2.3) of $\widehat{\sigma}(\alpha_i)$ is reached for some value s_i; then we can choose $\tilde{u}_\varepsilon$ so that $\tilde{u}_\varepsilon \in [\alpha_1, \alpha_2] \cup [s_1, s_2]$)

From (4.28), we get $\tilde{u}_\varepsilon = \alpha_i$ on Σ^i_η so that we can modify $\tilde{u}_\varepsilon$ inside Σ^i_η using the lemma 4.3 (ii). Let us consider a covering of $\Sigma^1_\eta \cup \Sigma^2_\eta$ by cells of size η by setting:

$$Q^k_\eta = \eta(k+Q) \ , \quad k \in \mathrm{Z}^{N-1}$$
$$I^i_\eta = \{k \in \mathrm{Z}^{N-1} \ ; \quad Q^k_\eta \subset \Sigma^i_\eta\}$$

To simplify, we assume that $\Sigma^i_\eta = \sum_{k \in I^i_\eta} Q^k_\eta$

By an easy rescaling, lemma 4.3 (ii) leads to the existence for each $k \in I^i_\eta$ of a sequence (w^k_ε) such that:

$$w^k_\varepsilon \longrightarrow \alpha_i \text{ in } L^2(Q^k_\eta) \ , \quad w^k_\varepsilon = \alpha_i \text{ on } \partial Q^k_\eta \cap \{x_N > 0\} \tag{4.31}$$
$$\lim_{\varepsilon \longrightarrow 0} F_\varepsilon(w^k_\varepsilon, Q^k_\eta) = \widehat{\widehat{\sigma}}(\alpha_i) \ \eta^{N-1}$$

Define:

$$u_\varepsilon = \begin{cases} \tilde{u}_\varepsilon & \text{on} \quad \Omega_\varepsilon \backslash \Sigma^1_\eta \cup \Sigma^2_\eta \\ w^k_\varepsilon & \text{on} \quad Q^k_\eta \ , \quad k \in I^1_\eta \cup I^2_\eta \\ 0 & \text{on} \quad \Omega \backslash \Omega_\varepsilon \end{cases}$$

It is easy to check that $u_\varepsilon \longrightarrow u$ in $L^2(\Omega)$. Moreover:

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \overline{\Omega_\varepsilon}) \leq F_\varepsilon(\tilde{u}_\varepsilon, \Omega^{-\eta} \cup \Gamma' \backslash \Gamma_\varepsilon) + F_\varepsilon(\tilde{u}_\varepsilon, \Gamma_\varepsilon \backslash (\Sigma^1_\eta \cup \Sigma^2_\eta)) + \\ + \sum_{k \in I^1_\eta \cup I^2_\eta} F_\varepsilon(w^k_\varepsilon, Q^k_\eta) \end{aligned} \tag{4.32}$$

From (4.27), $\sigma(\tilde{u}_\varepsilon)$ is bounded. Thus from (4.29), (4.31) and (4.32):

$$\limsup_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon, \overline{\Omega_\varepsilon}) \leq l_\eta + o(\eta) + \int_{\Gamma^1_\eta \cup \Gamma^2_\eta} \widehat{\widehat{\sigma}}(u^+) \ dH^{\mathrm{N}-1} \tag{4.33}$$

We conclude by passing to the limit as $\eta \longrightarrow 0$ using (4.30) and the fact that $\int_{\Gamma_0} |D\tilde{u}| = 0$ due to $H^{\mathrm{N}-1}(\partial A \cap \Gamma_0) = 0$.

Finally, in order to fit the constraint on the total mass ($\int_\Omega u_\varepsilon \ dx = \int_\Omega u \ dx$), we use Modica's method which consists in changing u_ε slightly inside one of the two phases (see [2]). $\square$

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