Asymptotics of a non-planar beam in linear elasticity

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Abstract

We study the asymptotic behavior of a linear elastic material lying in a thin tubular neighbourhood of a non planar line when the diameter of the section tends to zero. We first estimate the Korn constant in such a domain then we prove the convergence of the three dimensional model to a one dimensional model. This convergence is established in the framework of Γ-convergence. The resulting model is the one classically used in mechanics. It corresponds to a non-extensional line subjected to flexion and torsion. The torsion is an internal parameter which can eventually by eliminated but this elimination leads to a non-local energy. Indeed the non-planar geometry of the line couples the flexion and torsion terms.

keywords: Beam, Rod, Linear Elasticity, 3D-1D, Γ-convergence.

1 Introduction

Once this work was concluded we were advised that a similar study have been performed by G. Griso [2]. We also notice that our way to get an estimation for the Korn constant of the considered domain is similar to the way followed by Mora and Muller [3] to estimate the Rigidity constant of a straight rod in the framework of non linear elasticity. It should then be possible to extend the results presented here to the case of non linear materials.

2 The main result

2.1 Description of the beam

First let us define the mean line of the considered beam. Let $\mathcal{L}$ be a curve (a regular one dimensional manifold) in the physical space $\mathbb{R}^3$ and let $\varphi \in C^3([0,\ell],\mathbb{R}^3)$ be a curvilinear parametrization of $\mathcal{L}$.

For any $x_1 \in [0, \ell]$, we denote $t(x_1) := \varphi'(x_1)$ the unit vector tangent to the curve. We complete $t(x_1)$ in order to get an orthonormal basis $(t, n, b)(x_1)$. There are many choices for such a basis: though our notations are those usually used for the Frenet basis we emphasize that the position of $(n(x_1), b(x_1))$ in the plane orthogonal to $t(x_1)$ is free. We only assume that n and b, like t, belong to $C^2([0, \ell], \mathrm{I\!R}^3)$. We can then introduce the three functions τ, ξ, ρ in $C^1([0, \ell], \mathrm{I\!R})$ such that

$$\begin{aligned} t'(x_1) &= \tau(x_1)\, n(x_1) + \xi(x_1)\, b(x_1) \\ n'(x_1) &= -\tau(x_1)\, t(x_1) + \rho(x_1\,)b(x_1) \\ b'(x_1) &= -\xi(x_1)\, t(x_1) - \rho(x_1)\, n(x_1) \end{aligned} \tag{1}$$

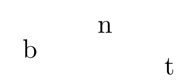

Figure 1: the mean line $\mathcal{L}$ of the beam, a non planar curve.

Note that the curvature of the line can be easily recognize as $\sqrt{\tau^2 + \xi^2}$ and the torsion of the line as $\rho + \dfrac{\tau\xi' - \xi\tau'}{\tau^2 + \xi^2}$.

Now let us describe the section of the considered beam. Let ω be a piecewise C^1 domain in $\mathrm{I\!R}^2$. It is a bounded simply-connected open set. Without loss of generality we can assume that 0 is the inertial center of ω and that the axis $\{0\} \times \mathrm{I\!R}$ and $\mathrm{I\!R} \times \{0\}$ are the principal inertial axis of ω :

$$\int_\omega x_2 dx_2\, dx_3 = 0, \quad \int_\omega x_3 dx_2\, dx_3 = 0, \quad \int_\omega x_2 x_3 dx_2\, dx_3 = 0 \tag{2}$$

x_3 x_3 x_2

x_2 x_1

Figure 2: the prototype section ω and the reference cylinder C.

We denote C the cylinder of $\mathbb{R}^3$: $C := [0, \ell] \times \omega$ and ϕ_ε the application defined by

$$\phi_\varepsilon(x_1, x_2, x_3) := \varphi(x_1) + \varepsilon(x_2\, n(x_1) + x_3\, b(x_1)). \tag{3}$$

Throughout this paper ε denotes a sequence tending to zero. For ε small enough, ϕ_ε is a C^2-diffeomorphism from C onto its image denoted Ω_ε (cf. figure 3). In the sequel the set Ω_ε will be refered as "the beam" and we will use in it the parametrization ϕ_ε.

Figure 3: the beam Ω_ε at rest.

The effect of a particular choice for the basis $(n(x_1), b(x_1))$ is now clear : we can tune the way the section turns around the mean line. This is a type of "torsion" of the beam which should not be confused with the torsion of the line $\mathcal{L}$ nor with the mechanical torsion which can result from the displacement of the beam. We emphasize the fact that, in figure 3, the beam is at rest.

2.2 Elastic energies

2.2.1 3-D linear elastic energy

Our goal is to study the behaviour of the beam Ω_ε in the framework of linear elasticity. We assume that the beam is fixed on its basis $\{x_1 = 0\}$, so any displacement field $\mathbf{u}$ has to vanish when $x_1 = 0$. The space of admissible displacements $\mathbf{u}$ is denoted :

$$H^1_b(\Omega_\varepsilon) := \{\mathbf{u} \in H^1(\Omega_\varepsilon, \mathbb{R}^3);\ \mathbf{u} = 0 \text{ when } x_1 = 0\}.$$

The strain tensor $\mathbf{e}(\mathbf{u})$ is the symmetric part of the gradient of $\mathbf{u}$ ($2\mathbf{e}(\mathbf{u}) := \nabla\mathbf{u} + (\nabla\mathbf{u})^t$). The elastic energy is a non-degenerated quadratic fonction of the strain tensor. We assume for sake of simplicity that the considered material is homogeneous and isotropic. The elastic energy $\mathbf{E}_\varepsilon$ is then characterized by the two Lamé coefficients λ, μ with $\mu > 0$ and $3\lambda + 2\mu > 0$:

$$\mathbf{E}_\varepsilon(\mathbf{u}) := \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}\left(\mu\|\mathbf{e}(\mathbf{u})\|^2 + \frac{\lambda}{2}(tr(\mathbf{e}(\mathbf{u})))^2\right)d\mathbf{x}. \tag{4}$$

This energy is defined on $H^1_b(\Omega_\varepsilon)$. It is naturally extended on $L^2(\Omega_\varepsilon, \mathbb{R}^3)$ by setting $\mathbf{E}_\varepsilon(\mathbf{u}) := +\infty$ if $\mathbf{u}$ does not belong to $H^1_b(\Omega_\varepsilon)$.

The scaling ε^{-2} is needed, as we will see later, to obtain a finite energy when passing to the limit. From the mechanical point of view this scaling can be interpreted as a choice for the force unit adapted to the weak rigidity of such a thin structure.

2.2.2 The limit one dimensional model

The limit energy we obtain is the one classically used in mechanics. Let us describe it: it is a one dimensional model for the line $\mathcal{L}$. The displacement is described by a vector field $\mathbf{u}$ on $\mathcal{L}$ but the mechanical description is made easier by the introduction of an extra scalar field $\mathbf{r}$ on $\mathcal{L}$. In mechanics $\mathbf{r}$ is interpreted as a measure of the rotation of the sections of the beam around the mean line. The space of admissible displacements is

$$H^2_b(\mathcal{L}) := \{\mathbf{u} \in H^2(\mathcal{L}, \mathbb{R}^3);\ \mathbf{u}' \cdot t = 0 \text{ along } \mathcal{L};\ \mathbf{u} = \mathbf{u}' = 0 \text{ when } x_1 = 0\}$$

while the space of admissible rotations is

$$H^1_b(\mathcal{L}) := \{\mathbf{r} \in H^1(\mathcal{L}, \mathbb{R});\ \mathbf{r} = 0 \text{ when } x_1 = 0\}.$$

Here $\mathcal{L}$ is endowed with the one dimensional Hausdorff measure and the derivatives are relative to the curvilinear abscissa. The elastic energy $\mathbf{F}$ is then characterized by a field of positive symmetric matrices A :

$$\mathbf{F}(\mathbf{u}, \mathbf{r}) := \frac{1}{2}\int_{\Omega_\varepsilon}(A \cdot (t \wedge \mathbf{u}' + \mathbf{r}t)') \cdot (t \wedge \mathbf{u}' + \mathbf{r}t)')\, d\mathcal{H}^1. \tag{5}$$

This energy is defined on $(H^2_b(\mathcal{L}) \times H^1_b(\mathcal{L}))$ but we extend it on $L^2(\mathcal{L}, \mathbb{R}^3 \times \mathbb{R})$ by setting $\mathbf{F}(\mathbf{u}, \mathbf{r}) := +\infty$ when $(\mathbf{u}, \mathbf{r})$ does not belong to the admissible space. Note that, due to the condition $\mathbf{u}' \cdot t = 0$ in the definition of $H^2_b(\mathcal{L})$, the line is non-extensional.

In terms of the displacement only, the energy is non local : it reads

$$\widetilde{\mathbf{F}}(\mathbf{u}) := \min_{r\in H^1_b(\mathcal{L})} \mathbf{F}(\mathbf{u},\mathbf{r}). \tag{6}$$

The matrix field A is related to the geometry of the section and to the material properties by

$$A(\varphi(x_1)) := \mu G\, t(x_1)\otimes t(x_1) + EI_2\, n(x_1)\otimes n(x_1) + YI_3\, b(x_1)\otimes b(x_1) \tag{7}$$

where Y is the Young modulus of the material $Y := \mu(3\lambda+2\mu)(\lambda+\mu)^{-1}$, I_2 and I_3 are the inertial moments of the section

$$I_2 := \int_\omega (x_3)^2 dx_2\, dx_3, \quad I_3 := \int_\omega (x_2)^2 dx_2\, dx_3 \tag{8}$$

and G is the infimum

$$G := \min\left\{ \int_\omega ((\frac{\partial\psi}{\partial x_3}+x_2)^2 + (\frac{\partial\psi}{\partial x_2}-x_3)^2)\, dx_2\, dx_3;\ \psi\in H^1(\omega,\mathrm{I\!R}^2)\right\} \tag{9}$$

which depends only on the geometry of the section. Therefore $\mathbf{F}$ reads

$$\mathbf{F}(\mathbf{u},\mathbf{r}) = \frac{1}{2}\int_{\Omega_\varepsilon} \Big(\mu G\,(t\wedge\mathbf{u}'+\mathbf{r}t)'\cdot t)^2 + YI_2\,(t\wedge\mathbf{u}'+\mathbf{r}t)'\cdot n)^2 + \\ + YI_3\,(t\wedge\mathbf{u}'+\mathbf{r}t)'\cdot b)^2\Big)\, d\mathcal{H}^1. \tag{10}$$

2.3 The main result

Let us first fix some notations. When no confusion can arise, we simply denote $|\mathcal{D}|$ the (Lebesgues or Hausdorff) measure of a set $\mathcal{D}$: in particular $|\Omega_\varepsilon|$, $|\omega|$, $|\mathcal{L}|$ denote respectively $\mathcal{H}^3(\Omega_\varepsilon$, $\mathcal{H}^2(\omega)$ and $\mathcal{H}^1(\mathcal{L})$. In the same way we omitt to precise the measure when invoking the mean values: for instance $⨍_\omega \psi$ denotes $|\omega|^{-1}\int_\omega \psi d\mathcal{H}^2$.

We denote $\omega(x_1)$ the section of Ω_ε defined by $\omega(x_1) := \Phi_\varepsilon(\{x_1\}\times\omega)$ and, for any $\mathbf{u}\in L^2(\Omega_\varepsilon)$ we denote $\bar{\mathbf{u}}\in L^2(\mathcal{L})$ its mean value on each section : $\bar{\mathbf{u}}(\varphi(x_1)) := ⨍_{\omega(x_1)} \mathbf{u}$.

Theorem 1 *(i) If $(\mathbf{u}_\varepsilon)$ is a sequence in $L^2(\Omega_\varepsilon,\mathrm{I\!R}^3)$ with bounded energy ($\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) < M$), then there exists a subsequence still denoted $(\mathbf{u}_\varepsilon)$ such that $\bar{\mathbf{u}}_\varepsilon$ converges in $L^2(\mathcal{L},\mathrm{I\!R}^3)$.*

(ii) For any sequence $(\mathbf{u}_\varepsilon)$ in $L^2(\Omega_\varepsilon,\mathrm{I\!R}^3)$ such that $\bar{\mathbf{u}}_\varepsilon$ converges to $\mathbf{u}$ in $L^2(\mathcal{L})$, we have

$$\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \geq \widetilde{\mathbf{F}}(\mathbf{u}) \tag{11}$$

(iii) For any $\mathbf{u}$ in $L^2(\mathcal{L},\mathrm{I\!R}^3)$, there exists a sequence $(\mathbf{u}_\varepsilon)$ in $L^2(\Omega_\varepsilon,\mathrm{I\!R}^3)$ such that $\bar{\mathbf{u}}_\varepsilon$ converges to $\mathbf{u}$ in $L^2(\mathcal{L},\mathrm{I\!R}^3)$ and

$$\limsup \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leq \widetilde{\mathbf{F}}(\mathbf{u}) \tag{12}$$

Remark 1 *We have decided to formulate this theorem in terms of the actual displacement fields, those which arise from the physical problem and are defined on Ω_ε and $\mathcal{L}$. One may prefer to refer to a fixed functionnal space. This is what is usually done in the study of straight beams [] and this is actually what we will do in the proof. Formulating the theorem in a fixed functionnal space has an important advantage : it can then be written in terms of Γ-convergence. A first disadvantage is that the choice of the fixed functional space is somehow arbitrary and one could then wonder whether the theorem is still valid for a different choice. A second disadvantage is the very intricate expression of the energy in the fixed space.*

Remark 2 *There is however a canonical way to reformulate the previous theorem in terms of Γ-convergence. Indeed let us associate to any function $\mathbf{u} \in L^2(\Omega_\varepsilon, \mathbb{R}^3)$ the vector valued measure $|\Omega_\varepsilon|^{-1}\mathbf{u}\, 1_{\Omega_\varepsilon} d\mathbf{x}$, where $|\Omega_\varepsilon|$ and 1_{Ω_ε} denote respectively the Lebesgue measure and the characteristic function of Ω_ε. In the same way let us associate to any $\mathbf{u} \in L^2(\mathcal{L}, \mathbb{R}^3)$ the vector valued measure $|\mathcal{L}|^{-1}\mathbf{u}\, d\mathcal{H}^1_{|\mathcal{L}}$, where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure. Let us endow the space of such vector valued measures with the weak* topology. A slightly different version of the previous theorem states the relative compactness of sequences with bounded energy and the Γ-convergence of $\mathbf{E}_\varepsilon$ to $\widetilde{\mathbf{F}}$. Indeed it is easy to check that the convergence of $\mathbf{u}_\varepsilon$ to $\mathbf{u}$ in the sense of measures implies, when the energy is bounded, the convergence of $\bar{\mathbf{u}}_\varepsilon$ to $\mathbf{u}$ in $L^2(\mathcal{L}, \mathbb{R}^3)$.*

Remark 3 *Let f be a continuous field of forces. A property of Γ-convergence (for details about the definition and the properties of Γ-convergence the reader can refer to [1]) shows that Theorem 1 remains valid when adding in $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$ and $\widetilde{\mathbf{F}}(\mathbf{u})$ respectively $-\fint_{\Omega_\varepsilon} f \cdot \mathbf{u}_\varepsilon$ and $-\fint_{\mathcal{L}} f \cdot \mathbf{u}$[1]. A second property of Γ-convergence shows that a sequence of equilibrium displacements for the beam (i.e. of minimizers of $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) - \fint_{\Omega_\varepsilon} f \cdot \mathbf{u}_\varepsilon$) converges to an equilibrium solution for the line $\mathcal{L}$ (i.e. a minimizer of $\widetilde{\mathbf{F}}(\mathbf{u}) - \fint_{\mathcal{L}} f \cdot \mathbf{u}$).*

2.4 Expression of energies in term of components

2.4.1 Expression of the beam energy in a fixed functional space

In order to work on a fixed functional space, we use the diffeomorphism Φ_ε as a change of variables which associates to any displacement field $\mathbf{u}_\varepsilon \in H^2_b$, using a slightly different typography, the vector field $u_\varepsilon := \mathbf{u}_\varepsilon \circ \Phi_\varepsilon$ defined on C. The space of admissible displacements u_ε becomes

$$H^1_b(C) := \{u \in H^1(C, \mathbb{R}^3);\ u(0, x_2, x_3) = 0,\ \forall (x_2, x_3) \in \omega\}. \tag{13}$$

In the same way we associate to the strain tensor $\mathbf{e}(\mathbf{u}_\varepsilon)$ the tensor field e_ε defined on C by $e_\varepsilon = \mathbf{e}(\mathbf{u}_\varepsilon) \circ \Phi_\varepsilon$. Note that e_ε is no more the symmetric part of the gradient of u_ε. Let us establish the relation which links e_ε and u_ε.

[1] Different choices of forces could be considered as in [4] to the price of a different formulation of the theorem.

As usual when using curvilinear coordinates, we introduce for any parameter $x = (x_1, x_2, x_3)$ the natural basis $(g_{\varepsilon 1}(x), g_{\varepsilon 2}(x), g_{\varepsilon 3}(x))$ defined by

$$g_{\varepsilon i}(x) = \frac{\partial \phi_\varepsilon}{\partial x_i}, \quad \forall i \in \{1, 2, 3\}.$$

The explicit computation of this basis leads to

$$\begin{aligned} g_{\varepsilon 1}(x) &= j_\varepsilon(x)\, t(x_1) + \varepsilon \rho(x_1)(x_2\, b(x_1) - x_3\, n(x_1)), \\ g_{\varepsilon 2}(x) &= \varepsilon n(x_1), \qquad\qquad g_{\varepsilon 3}(x) = \varepsilon b(x_1), \end{aligned} \tag{14}$$

where $\varepsilon^2 j_\varepsilon(x)$ is the jacobian of the diffeomorphism :

$$j_\varepsilon(x) = 1 - \varepsilon(x_2 \tau(x_1) + x_3 \xi(x_1)). \tag{15}$$

This basis is not orthogonal. So it is useful to introduce the dual basis $(g_\varepsilon^1, g_\varepsilon^2, g_\varepsilon^3)$ defined by $g_\varepsilon^i(x) \cdot g_{\varepsilon j}(x) = \delta_j^i$ (where δ_j^i is the Kronecker symbol) and the metric tensor $\mathcal{G}_\varepsilon^{ij} := g_\varepsilon^i \cdot g_\varepsilon^j$. We have :

$$\begin{aligned} g_\varepsilon^1(x) &= \frac{1}{j_\varepsilon(x)} t(x_1), \quad g_\varepsilon^2(x) = \frac{1}{\varepsilon} n(x_1) + \frac{\rho(x_1) x_3}{j_\varepsilon(x)} t(x_1), \\ g_\varepsilon^3(x) &= \frac{1}{\varepsilon} b(x_1) - \frac{\rho(x_1) x_2}{j_\varepsilon(x)} t(x_1). \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathcal{G}_\varepsilon^{11}(x) &= \frac{1}{(j_\varepsilon(x))^2}, \qquad \mathcal{G}_\varepsilon^{12}(x) = \frac{\rho(x_1) x_3}{(j_\varepsilon(x))^2}, \\ \mathcal{G}_\varepsilon^{22}(x) &= \frac{1}{\varepsilon^2} + \left(\frac{\rho(x_1) x_3}{j_\varepsilon(x)}\right)^2, \quad \mathcal{G}_\varepsilon^{13}(x) = \frac{-\rho(x_1) x_2}{(j_\varepsilon(x))^2}, \\ \mathcal{G}_\varepsilon^{23}(x) &= \frac{-(\rho(x_1))^2 x_2 x_3}{(j_\varepsilon(x))^2}, \qquad \mathcal{G}_\varepsilon^{33}(x) = \frac{1}{\varepsilon^2} + \left(\frac{\rho(x_1) x_2}{j_\varepsilon(x)}\right)^2, \end{aligned} \tag{17}$$

The components of u_ε or e_ε in the dual of the natural basis are denoted respectively $u_{\varepsilon i} := u_\varepsilon \cdot g_{\varepsilon i}$ and $e_{\varepsilon ij} = (e_\varepsilon \cdot g_{\varepsilon j}) \cdot g_{\varepsilon i}$. We have $u_\varepsilon = u_{\varepsilon i}\, g_\varepsilon^i$ and $e_\varepsilon = e_{\varepsilon ij}\, g_\varepsilon^i \otimes g_\varepsilon^j$.

Note that, troughout this paper, we make use of the Einstein summation convention (any repeated indices in a product have to be summed from 1 to 3).

To compute any spatial derivative using the curvilinear coordinates it is necessary to make explicit the Christoffel symbols $\Gamma_{\varepsilon ij}^k = \Gamma_{\varepsilon ji}^k := g_\varepsilon^k \cdot \frac{\partial}{\partial x_i}(g_{\varepsilon j})$:

$$\begin{aligned} \Gamma_{\varepsilon 11}^1(x) &= -\frac{\varepsilon}{j_\varepsilon(x)} \Big(x_2 \tau'(x_1) + x_3 \xi'(x_1) + \rho(x_1)(x_2 \xi(x_1) - x_3 \tau(x_1)) \Big), \\ \Gamma_{\varepsilon 11}^2(x) &= \frac{j_\varepsilon(x)}{\varepsilon} \tau(x_1) - \rho'(x_1) x_3 - (\rho(x_1))^2 x_2 + \rho(x_1) x_3 \Gamma_{\varepsilon 11}^1(x), \\ \Gamma_{\varepsilon 11}^3(x) &= \frac{j_\varepsilon(x)}{\varepsilon} \xi(x_1) + \rho'(x_1) x_2 - (\rho(x_1))^2 x_3 - \rho(x_1) x_2 \Gamma_{\varepsilon 11}^1(x), \end{aligned}$$

$$\Gamma^1_{\varepsilon 12}(x) = -\frac{\varepsilon\tau(x_1)}{j_\varepsilon(x)}, \qquad \Gamma^2_{\varepsilon 12} = -\frac{\varepsilon\tau(x_1)\rho(x_1)x_3}{j_\varepsilon(x)},$$
$$\Gamma^3_{\varepsilon 12}(x) = \rho(x_1)(1+\frac{\varepsilon\tau(x_1)x_2}{j_\varepsilon(x)}), \qquad \Gamma^1_{\varepsilon 13}(x) = -\frac{\varepsilon\xi(x_1)}{j_\varepsilon(x)},$$
$$\Gamma^2_{\varepsilon 13}(x) = -\rho(x_1)(1+\frac{\varepsilon\xi(x_1)x_3}{j_\varepsilon(x)}), \qquad \Gamma^3_{\varepsilon 13}(x) = \frac{\varepsilon\xi(x_1)\rho(x_1)x_2}{j_\varepsilon(x)},$$
$$\Gamma^i_{\varepsilon 22}(x) = \Gamma^i_{\varepsilon 33}(x) = \Gamma^i_{\varepsilon 23}(x) = 0, \quad \forall i \in \{1,2,3\} \tag{18}$$

The relation between e_ε and u_ε is then

$$2e_{\varepsilon ij}(u_\varepsilon) = \frac{\partial u_{\varepsilon i}}{\partial x_j} + \frac{\partial u_{\varepsilon j}}{\partial x_i} - 2\Gamma^k_{\varepsilon ij} u_{\varepsilon k} \tag{19}$$

Throughout the change of variables, the functional $\mathbf{E}_\varepsilon$ is transformed in

$$E_\varepsilon(u) := \frac{1}{\varepsilon^2}\int_C \left(\mu\|e_\varepsilon(u)\|^2 + \frac{\lambda}{2}(tr(e_\varepsilon(u)))^2\right)(x)\,\frac{1}{j_\varepsilon(x)}\,dx. \tag{20}$$

To explicit the functional E_ε, we need to compute $\|e_\varepsilon\|^2$ and $tr(e_\varepsilon)$ in term of the components of e_ε. We get

$$\|e_\varepsilon\|^2 = \|e_{\varepsilon ij}\, g^i_\varepsilon \otimes g^j_\varepsilon\|^2 = e_{\varepsilon ij}e_{\varepsilon kl}\,\mathcal{G}^{il}_\varepsilon\mathcal{G}^{jk}_\varepsilon,$$
$$(tr(e_\varepsilon))^2 = (e_{\varepsilon ij}\, tr(g^i_\varepsilon \otimes g^j_\varepsilon))^2 = e_{\varepsilon ij}e_{\varepsilon kl}\,\mathcal{G}^{ij}_\varepsilon\mathcal{G}^{kl}_\varepsilon.$$

Denoting

$$R^{ijkl}_\varepsilon := 2\mu\mathcal{G}^{il}_\varepsilon\mathcal{G}^{jk}_\varepsilon + \lambda\mathcal{G}^{ij}_\varepsilon\mathcal{G}^{kl}_\varepsilon \tag{21}$$

(which is nothing else but the rigidity tensor expressed in the natural basis), we have

$$E_\varepsilon(u) = \frac{1}{2\varepsilon^2}\int_C R^{ijkl}_\varepsilon e_{\varepsilon ij}e_{\varepsilon kl}\,\frac{1}{j_\varepsilon(x)}\,dx. \tag{22}$$

It is not very attractive at this point to replace R_ε using (21) where the $\mathcal{G}^{ij}_\varepsilon$ are given by (18) and to replace $e_{\varepsilon ij}$ using (19) in which the symbols $\Gamma^k_{\varepsilon ij}$ are given by (18). It is clear that the expression of E_ε is very intricate!

2.4.2 Expression of the limit energy in terms of components

In a natural way, we associate to any $(\mathbf{u},\mathbf{r}) \in L^2(\mathcal{L},\mathrm{I\!R}^4)$ a field $(u,r) \in L^2([0,L],\mathrm{I\!R}^4)$ by setting $u = \mathbf{u}\circ\varphi$ and $r = \mathbf{r}\circ\varphi$. We denote (u_1,u_2,u_3) the components of u in the basis (t,n,b). The explicit computation of the components of $(t\wedge\mathbf{u}' + \mathbf{r}t)'$ is straightforward and the functional $\mathbf{F}$ is transformed in

$$F(u,r) = \frac{1}{2}\int_0^L \left(\mu G\, q_1^2 + YI_2\, q_2^2 + YI_3\, q_3^2\right)\,dx_1 \tag{23}$$

where

$$q_1 := r' + \tau u_3' - \xi u_2' + \rho(\tau u_2 + \xi u_3), \tag{24}$$
$$q_2 := \tau r - u_3'' - 2\rho u_2' - (\xi' + \rho\tau)u_1 - (\rho' + \xi\tau)u_2 - (\xi^2 - \rho^2)u_3, \tag{25}$$
$$q_3 := \xi r + u_2'' - 2\rho u_3' - (-\tau' + \rho\xi)u_1 - (\rho^2 - \tau^2)u_2 - (\rho' - \tau\xi)u_3. \tag{26}$$

It is also useful to note that the non-extension condition $\mathbf{u}' \cdot t = 0$ reads

$$u_1' - \tau u_2 - \xi u_3 = 0. \tag{27}$$

3 Korn inequality

The first thing to study is the asymtotic behavior of the Korn constant in Ω_ε. In the case of straight beams, this is usually done by comparing the expression for the energy (22) (when $\mu = 1$ and $\lambda = 0$) and the integral of the symmetric part of the gradient of u and then applying Korn inequality in C. In the straight case [], it becomes clear that, for ε small enough, and for any matrix e, $R_\varepsilon^{ijkl} e_{ij} e_{kl}$ is larger that $e_{ij}e_{ij}$. This could be also obtained (with more work) in our case. The principal difficulty lies elsewhere: $e_\varepsilon(u_\varepsilon)$ is no more the symmetric part of u_ε. Instead u_ε appears directly in the expression (19) of e_ε. Still more serious is the fact that, in this expression, some of the Christoffel symbols are very large (of order ε^{-1}) : a control on e_ε does not provide any control on the symmetric part of the gradient of u_ε. Let us then study the asymtotic behavior of the Korn constant using a different direct method.

We establish a first lemma which states that two almost identical domains have almost the same Korn constant. For any $u \in H^1(\Omega, \mathrm{I\!R}^N)$, we denote $e(u)$ the symmetric part of the gradient of u and $\nabla^a u := \nabla u - e(u)$ its skew-symmetric part. A variant of Korn inequality [] states, for any domain[2] Ω of $\mathrm{I\!R}^N$, the existence of a constant K_Ω such that, for any $u \in H^1(\Omega, \mathrm{I\!R}^N)$,

$$\int_\Omega \|\nabla^a u - (⨍_\Omega \nabla^a u)\|^2 \, d\mathbf{x} \leq K_\Omega^2 \int_\Omega \|e(u)\|^2 \, dx, \tag{28}$$

where $⨍_\Omega \nabla^a u$ denotes the mean value of $\nabla^a u$: $⨍_\Omega \nabla^a u := |\Omega|^{-1} \int_\Omega \nabla^a u \, dx$. In the following K_Ω denotes the smallest constant satisfying the previous inequality: in other words

$$K_\Omega := \sup \left\{ \|\nabla^a u\|_{L^2(\Omega)};\ u \in H^1(\Omega, \mathrm{I\!R}^N),\ \int_\Omega \nabla^a u dx = 0,\ \int_\Omega \|e(u)\|^2 \, dx \leq 1 \right\}$$

Lemma 1 *Let $(\mathcal{D}_\varepsilon)$ be a sequence of domains in $\mathrm{I\!R}^N$. Assume that there exist a domain $\mathcal{D}$ and , for any ε, a diffeomorphism Ψ_ε from $\mathcal{D}_\varepsilon$ onto $\mathcal{D}$ satisfying, at every point $x \in \mathcal{D}_\varepsilon$, $\|\nabla \Psi_\varepsilon(x) - Id\| \leq \varepsilon$. Then there exists a constant $c > 0$, depending only on N and $\mathcal{D}$, such that for ε small enough,*

$$K_{\mathcal{D}_\varepsilon} \leq K_{\mathcal{D}}(1 + c\,\varepsilon) \tag{29}$$

Proof : Let $u \in H^1(\mathcal{D}_\varepsilon, \mathrm{I\!R}^N)$ satisfying

$$\int_{\mathcal{D}_\varepsilon} \|e(u)\|^2 \, dx \leq 1 \qquad \text{and} \qquad \int_{\mathcal{D}_\varepsilon} \nabla^a u \, dx = 0 \tag{30}$$

[2] Here we call “domain” a piecewise-C^1, bounded and connected open set.

Let us define $v := u \circ \Psi_\varepsilon^{-1} \in H^1(\mathcal{D}, \mathbb{R}^N)$, we have

$$u = v \circ \Psi_\varepsilon, \quad \nabla u(x) = \nabla v(\Psi_\varepsilon(x)) \cdot \nabla \Psi_\varepsilon(x).$$

Hence

$$\|\nabla u(x) - \nabla v(\Psi_\varepsilon(x))\| = \|\nabla v(\Psi_\varepsilon(x)) \cdot (\nabla \Psi_\varepsilon(x) - Id)\| \le \varepsilon \|\nabla v(\Psi_\varepsilon(x))\|. \quad (31)$$

which gives immediately the same estimation for the symmetric and skew symmetric parts $e(u)(x) - e(v)(\Psi_\varepsilon(x))$ and $\nabla^a u(x) - \nabla^a v(\Psi_\varepsilon(x))$.

On the other hand, setting $c_1 := \sqrt{N} + 1$, we have, for ε small enough and for any $x \in \mathcal{D}_\varepsilon$,

$$1 - c_1\varepsilon \le det(\nabla \Psi_\varepsilon(x)) \le 1 + c_1\varepsilon, \quad \text{and } 1 - c_1\varepsilon \le det(\nabla \Psi_\varepsilon^{-1}(x)) \le 1 + c_1\varepsilon.$$

So for any function $\psi \in L^1(\mathcal{D}, \mathbb{R}^m)$

$$\| \int_{\mathcal{D}} \psi(y)dy - \int_{\mathcal{D}_\varepsilon} \psi(\Psi_\varepsilon(x))dx\| \le c_1\varepsilon \int_{\mathcal{D}_\varepsilon} \|\psi(\Psi_\varepsilon(x))\| \, dx, \quad (32)$$

or

$$\| \int_{\mathcal{D}} \psi(y)dy - \int_{\mathcal{D}_\varepsilon} \psi(\Psi_\varepsilon(x))dx\| \le c_1\varepsilon \int_{\mathcal{D}} \psi(y) \, dy, \quad (33)$$

and

$$\| \fint_{\mathcal{D}} \psi(y)dy - \fint_{\mathcal{D}_\varepsilon} \psi(\Psi_\varepsilon(x))dx\| \le 2c_1\varepsilon \fint_{\mathcal{D}} \|\psi(\Psi_\varepsilon(x))\| \, dx, \quad (34)$$

or

$$\| \fint_{\mathcal{D}} \psi(y)dy - \fint_{\mathcal{D}_\varepsilon} \psi(\Psi_\varepsilon(x))dx\| \le 2c_1\varepsilon \fint_{\mathcal{D}} \psi(y) \, dy. \quad (35)$$

Applying inequality (34) with $\psi = \nabla^a v$, using (31), (30) and (35) we get

$$\begin{aligned}
\| \fint_{\mathcal{D}} \nabla^a v(y)dy\| \quad & \le \| \fint_{\mathcal{D}_\varepsilon} \nabla^a v(\Psi_\varepsilon(x))dx\| + 2c_1\varepsilon \fint_{\mathcal{D}_\varepsilon} \|\nabla^a v(\Psi_\varepsilon(x))\|dx \\
& \le \| \fint_{\mathcal{D}_\varepsilon} \nabla^a u(x)dx\| + (2c_1 + 1)\varepsilon \fint_{\mathcal{D}_\varepsilon} \|\nabla v(\Psi_\varepsilon(x))\|dx \\
& \le (2c_1 + 1)\varepsilon \fint_{\mathcal{D}_\varepsilon} \|\nabla v(\Psi_\varepsilon(x))\|dx \\
& \le (2c_1 + 2)\varepsilon \fint_{\mathcal{D}} \|\nabla v(y)\|dy.
\end{aligned}$$

Hence

$$\| \fint_{\mathcal{D}} \nabla^a v(y)dy\| \le (2c_1 + 2)|\mathcal{D}|^{-1/2}\varepsilon \|\nabla v\|_{L^2(\mathcal{D})}. \quad (36)$$

Applying now inequality (32) to $e(v)$, using (31), (30) and finally (33), we get

$$\begin{aligned}
\int_{\mathcal{D}} \|e(v)(y)\|^2\,dy \quad & \le (1+c_1\varepsilon) \int_{\mathcal{D}_\varepsilon} \|e(v)(\Psi_\varepsilon(x))\|^2\,dx \\
& \le (1+c_1\varepsilon) \int_{\mathcal{D}_\varepsilon} \left(\|e(u)(x)\| + \varepsilon\|\nabla(v)(\Psi_\varepsilon(x))\|\right)^2\,dx \\
& \le (1+c_1\varepsilon) \left(1+\varepsilon\left(\int_{\mathcal{D}_\varepsilon} \|\nabla v(\Psi_\varepsilon(x))\|^2\,dx\right)^{\frac{1}{2}}\right)^2 \\
& \le (1+c_1\varepsilon) \left(1+\varepsilon\left((1+c_1\varepsilon)\int_{\mathcal{D}} \|\nabla v(y)\|^2\,dy\right)^{\frac{1}{2}}\right)^2 .
\end{aligned}$$

Hence

$$\|e(v)\|_{L^2(\mathcal{D})} \le 1 + c_1\varepsilon + 2\varepsilon\|\nabla v\|_{L^2(\mathcal{D})}. \tag{37}$$

and Korn inequality in $\mathcal{D}$ leads to

$$\|\nabla^a v - \fint_{\mathcal{D}} \nabla^a v\|_{L^2(\mathcal{D})} \le K_{\mathcal{D}}(1 + c_1\varepsilon + 2\varepsilon\|\nabla v\|_{L^2(\mathcal{D})}) \tag{38}$$

Using (36) we obtain

$$\|\nabla^a v\|_{L^2(\mathcal{D})} \le K_{\mathcal{D}}(1+c_1\varepsilon) + (2K_{\mathcal{D}} + 2c_1 + 2)\,\varepsilon\,\|\nabla v\|_{L^2(\mathcal{D})} \tag{39}$$

and using again (37)

$$\|\nabla v\|_{L^2(\mathcal{D})} \le (K_{\mathcal{D}}+1)(1+c_1\varepsilon) + (2K_{\mathcal{D}} + 2c_1 + 4)\,\varepsilon\,\|\nabla v\|_{L^2(\mathcal{D})}. \tag{40}$$

Therefore $\|\nabla v\|_{L^2(\mathcal{D})}$ is bounded. Indeed, for ε small enough,

$$\|\nabla v\|_{L^2(\mathcal{D})} \le \frac{(K_{\mathcal{D}}+1)(1+c_1\varepsilon)}{1-(2K_{\mathcal{D}}+2c_1+4)\,\varepsilon} \le 2K_{\mathcal{D}} + 2. \tag{41}$$

Pluging this majoration in (39) we get

$$\|\nabla^a v\|_{L^2(\mathcal{D})} \le K_{\mathcal{D}} + c_2\,\varepsilon \tag{42}$$

where $c_2 := K_{\mathcal{D}}c_1 + (2K_{\mathcal{D}} + 2c_1 + 2)(2K_{\mathcal{D}} + 2)$. Using again (31) we have

$$\|\nabla^a u\|_{L^2(\mathcal{D}_\varepsilon)} \le \|\nabla^a v \circ \Psi_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} + \varepsilon\|\nabla v \circ \Psi_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)}, \tag{43}$$

and from (33)

$$\|\nabla^a v \circ \Psi_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} \le \sqrt{1+c_1\varepsilon}\;\|\nabla^a v\|_{L^2(\mathcal{D})}. \tag{44}$$

$$\|\nabla v \circ \Psi_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} \le \sqrt{1+c_1\varepsilon}\;\|\nabla v\|_{L^2(\mathcal{D})}. \tag{45}$$

Collecting (41), (42), (43), (44) and (45)

$$\|\nabla^a u\|_{L^2(\mathcal{D}_\varepsilon)} \le \sqrt{1+c_1\varepsilon}\,(K_{\mathcal{D}} + 2c_2\,\varepsilon) \le K_{\mathcal{D}}(1+c\varepsilon) \tag{46}$$

with $c := c_1 + 3c_2/K_{\mathcal{D}}$. □

As the Korn inequality (28) is invariant by rescaling we easily get the following corollary:

Corollary 1 *Lemma 1 remains valid if we only assume that each domain* $\mathcal{D}_\varepsilon$ *is nearly homothetic to* $\mathcal{D}$*. More precisely if, for any* ε*, there exist a real* k_ε *and a diffeomorphism* Ψ_ε *from* $\mathcal{D}_\varepsilon$ *onto* $\mathcal{D}$ *satisfying, at every point* $x \in \mathcal{D}_\varepsilon$*,* $\|\nabla\Psi_\varepsilon(x) - k_\varepsilon Id\| \leq \varepsilon$*.*

Now we can state a Korn theorem for Ω_ε.

Theorem 2 *There exists a constant* K *depending only on* ω *and* L *such that, for* ε *small enough and for any* $\mathbf{u}$ *in* $H^1_b(\Omega_\varepsilon)$*,*

$$\|\mathbf{u}\|_{H^1(\Omega_\varepsilon)} \leq \frac{K}{\varepsilon}\|\mathbf{e}(\mathbf{u})\|_{L^2(\Omega_\varepsilon)} \tag{47}$$

Proof : In order to take easily into account the boundary condition, let us extend (without changing the notations) the domain Ω_ε by considering a suitable exension of Φ_ε on $[-a_\varepsilon, L] \times \omega$ where a_ε is chosen in $[-2\varepsilon, -\varepsilon]$ in such a way that $\varepsilon^{-1}(L + a_\varepsilon)$ is an integer (denoted n_ε).

Let us also extend any $\mathbf{u} \in H^1_b(\Omega_\varepsilon)$ by setting $\mathbf{u} = 0$ on the new part $\{x_1 \leq 0\}$.

We split the domain in n_ε parts by defining, for any $i \in \{1, \ldots, n_\varepsilon\}$,

$$\Omega^i_\varepsilon := \widetilde{\Phi}_\varepsilon([a_\varepsilon + (i-1)\varepsilon, a_\varepsilon + i\varepsilon] \times \omega) \tag{48}$$

Defining h^i_ε by $h^i_\varepsilon(x_1, x_2, x_3) := (\varepsilon^{-1}(x_1 - a_\varepsilon) - (i-1), x_2, x_3)$, the application $h^i_\varepsilon \circ \Phi^{-1}_\varepsilon$ is a diffeormorphism from Ω^i_ε onto the cylinder $[0,1] \times \omega$. An explicit computation leads to $\|\nabla(\Phi_\varepsilon \circ (h^i_\varepsilon)^{-1} - \varepsilon Id| \leq \varepsilon d\sqrt{\tau^2 + \xi^2 + \rho^2}$ where d is the diameter of ω. We can then apply corollary 1 and use the same Korn constant for every part Ω^i_ε. Indeed, denoting $K_1 := K_{[0,1]\times\omega}$, we have for ε small enough and for any $i \in \{1, \ldots, n_\varepsilon\}$,

$$K_{\Omega^i_\varepsilon} \leq 2K_1 \tag{49}$$

In the same way $h^i_\varepsilon \circ \Phi^{-1}_\varepsilon$ is a diffeormorphism from $\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon$ onto the cylinder $[0,2] \times \omega$ and, denoting $K_2 := K_{[0,2]\times\omega}$, we have for ε small enough and for any $i \in \{1, \ldots, n_\varepsilon - 1\}$,

$$K_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \leq 2K_2 \tag{50}$$

Let us introduce the mean rotation of each part $r^i_\varepsilon := \fint_{\Omega^i_\varepsilon} \nabla^a \mathbf{u}$ and the piecewise constant function $r := \sum_{i=1}^{n_\varepsilon} r^i_\varepsilon 1_{\Omega^i_\varepsilon}$. Korn inequality on each $\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon$ gives

$$\int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \left\| \nabla^a \mathbf{u} - \fint_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \nabla^a \mathbf{u} \right\|^2 d\mathbf{x} \leq 4K_2^2 \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(\mathbf{u})\|^2 \, d\mathbf{x}$$

Restricting the integral at the right hand side of this inequality to Ω^i_ε, we get

$$|\Omega^i_\varepsilon| \fint_{\Omega^i_\varepsilon} \left\| \nabla^a \mathbf{u} - \fint_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \nabla^a \mathbf{u} \right\|^2 d\mathbf{x} \leq 4K_2^2 \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(\mathbf{u})\|^2 \, d\mathbf{x},$$

which implies

$$|\Omega_\varepsilon^i| \left\| r_\varepsilon^i - \fint_{\Omega_\varepsilon^i \cup \Omega_\varepsilon^{i+1}} \nabla^a \mathbf{u} \right\|^2 \leq 4K_2^2 \int_{\Omega_\varepsilon^i \cup \Omega_\varepsilon^{i+1}} \|e(\mathbf{u})\|^2 \, d\mathbf{x}.$$

In the same way:

$$|\Omega_\varepsilon^{i+1}| \left\| r_\varepsilon^{i+1} - \fint_{\Omega_\varepsilon^i \cup \Omega_\varepsilon^{i+1}} \nabla^a \mathbf{u} \right\|^2 \leq 4K_2^2 \int_{\Omega_\varepsilon^i \cup \Omega_\varepsilon^{i+1}} \|e(\mathbf{u})\|^2 \, d\mathbf{x}$$

From which we deduce (using the fact that for any i, $|\Omega_\varepsilon^i| > \varepsilon^3 |\omega|/2$)

$$\left\| r_\varepsilon^{i+1} - r_\varepsilon^i \right\|^2 \leq \frac{32K_2^2}{|\omega|\varepsilon^3} \int_{\Omega_\varepsilon^i \cup \Omega_\varepsilon^{i+1}} \|e(\mathbf{u})\|^2 \, d\mathbf{x}$$

Using the fact that $\mathbf{u} = 0$ on Ω_ε^1 and so that $r_\varepsilon^1 = 0$, we get

$$\begin{aligned}
\|r_\varepsilon^i\|^2 \qquad & \leq (i-1) \sum_{j=1}^{i-1} \|r_\varepsilon^{j+1} - r_\varepsilon^j\|^2 \\
& \leq n_\varepsilon \frac{32K_2^2}{|\omega|\varepsilon^3} \sum_{j=1}^{i-1} \int_{\Omega_\varepsilon^i \cup \Omega_\varepsilon^{i+1}} \|e(\mathbf{u})\|^2 \, d\mathbf{x} \\
& \leq n_\varepsilon \frac{64K_2^2}{|\omega|\varepsilon^3} \int_{\Omega_\varepsilon} \|e(\mathbf{u})\|^2 \, d\mathbf{x} \\
& \leq L \frac{128K_2^2}{|\omega|\varepsilon^4} \int_{\Omega_\varepsilon} \|e(\mathbf{u})\|^2 \, d\mathbf{x}
\end{aligned}$$

Thus (using the fact that for any i, $|\Omega_\varepsilon^i| < 2\varepsilon^3 |\omega|$)

$$\|r\|_{L^2(\Omega_\varepsilon)} \leq \frac{16K_2 L}{\varepsilon} \|e(\mathbf{u})\|_{L^2(\Omega_\varepsilon)} \tag{51}$$

Korn inequality on each Ω_ε^i reads

$$\int_{\Omega_\varepsilon^i} \left\| \nabla^a \mathbf{u} - r_\varepsilon^i \right\|^2 \, d\mathbf{x} \leq 4K_1^2 \int_{\Omega_\varepsilon^i} \|e(\mathbf{u})\|^2 \, d\mathbf{x},$$

and by summation we get

$$\|\nabla^a \mathbf{u} - r\|_{L^2(\Omega_\varepsilon)} \leq 2K_1 \, \|e(\mathbf{u})\|_{L^2(\Omega_\varepsilon)}. \tag{52}$$

This inequality together with (51) gives

$$\|\nabla^a \mathbf{u}\|_{L^2(\Omega_\varepsilon)} \leq \frac{K_3}{\varepsilon} \|e(\mathbf{u})\|_{L^2(\Omega_\varepsilon)}, \tag{53}$$

for any constant $K_3 > 16K_2L$, and therefore

$$\|\nabla \mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} \leq \left(1 + \frac{K_3^2}{\varepsilon^2}\right) \|e(\mathbf{u})\|^2_{L^2(\Omega_\varepsilon)}. \tag{54}$$

Let us finally check that the Poincaré constant in $H^1_b(\Omega_\varepsilon)$ is bounded. Indeed, let us use the change of variables Φ_ε given by (3). The associated jacobian $\varepsilon^2 j_\varepsilon$ is given by (15). We have, for ε small enough,

$$\|\nabla \Phi_\varepsilon\| \leq 2 \quad \text{and} \quad \frac{1}{2} \leq j_\varepsilon \leq 2$$

and, as L is a clear upperbound for the Poincaré constant on $H^1_b(C)$,

$$\begin{aligned}
\|\mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} \quad & \leq \int_C \|u(\Phi_\varepsilon(x)\|^2 \varepsilon^2\, j_\varepsilon\, dx \leq 2 \int_C \|u(\Phi_\varepsilon(x)\|^2 \varepsilon^2\, dx \\
& \leq 2L^2\varepsilon^2 \int_C \|\nabla(u \circ \Phi_\varepsilon)(x)\|^2\, dx \leq 8L^2\varepsilon^2 \int_C \|\nabla u(\Phi_\varepsilon(x))\|^2\, dx \\
& \leq 8L^2\varepsilon^2 \int_{\Omega_\varepsilon} \|\nabla u(\mathbf{x})\|^2 \frac{1}{\varepsilon^2 j_\varepsilon(\Phi_\varepsilon^{-1}(\mathbf{x})} \, d\mathbf{x} \\
& \leq 16L^2 \int_{\Omega_\varepsilon} \|\nabla u(\mathbf{x})\|^2\, d\mathbf{x}
\end{aligned} \tag{55}$$

Hence

$$\|\mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} \leq 16L^2 \left(\frac{K_3^2}{\varepsilon^2} + 1\right) \|e(\mathbf{u})\|^2_{L^2(\Omega_\varepsilon)}. \tag{56}$$

The theorem is proved, choosing $K > 4LK_3$. □

4 Proof of the main theorem

4.1 Compactness

First, let us extend (without changing the notations) the domain Ω_ε by considering a suitable extension of Φ_ε on $[-a, L] \times \omega$ (with $a > 0$) and extend any $\mathbf{u} \in H^1_b(\Omega_\varepsilon)$ by setting $\mathbf{u} = 0$ on the new part $\{x_1 \leq 0\}$.

It is clear that it is enough to consider in the proof of points (i) or (ii) of Theorem 1 only sequences $(\mathbf{u}_\varepsilon)$ with bounded energy ($\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leq M$). Moreover we can restrict our attention to a subsequence (still denote $(\mathbf{u}_\varepsilon)$) such that $\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \lim \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$. The statements will then be proved, when proved for some subsequence.

It is well known that, for any matrix A,

$$\mu\|A\|^2 + \frac{\lambda}{2}(tr(A)^2 \geq \eta\|A\|^2$$

where $\eta := \min\{\mu, \frac{2\mu+3\lambda}{2}\} > 0$. Therefore the asumption $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leq M$ implies

$$\|e(\mathbf{u}_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} \leq \eta^{-1} M \varepsilon^4, \tag{57}$$

and owing to Theorem 2,

$$\|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leq K^2 \eta^{-1} M \varepsilon^2, \tag{58}$$

In order to work in the fixed functional space, let us use the change of variables Φ_ε given by (3) and denote $u_\varepsilon = \mathbf{u}_\varepsilon \circ \Phi_\varepsilon^{-1}$. Computations similar to (55) , lead to

$$\|u_\varepsilon\|^2_{L^2(C)} \leq 2\varepsilon^{-2} \|\mathbf{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq 2K^2\eta^{-1}M,$$

and

$$\|\nabla u_\varepsilon\|^2_{L^2(C)} \leq 8\varepsilon^{-2} \|\nabla \mathbf{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq 8K^2\eta^{-1}M.$$

The sequence (u_ε) is bounded in $H^1(C)$ and, up to a subsequence[3], converges weakly to some u^0 in $H^1(C, \mathbb{R}^3)$. Then (u_ε) converges strongly to u^0 in $L^2(C)$, $(\bar{u}_\varepsilon)$ converges strongly to $\bar{u}^0$ in $L^2([0, L])$ and $(\bar{\mathbf{u}}_\varepsilon)$ converges strongly to $\mathbf{u} := \bar{u}^0 \circ \varphi^{-1}$ in $L^2(\mathcal{L})$. Point (i) is proved. □

4.2 Lowerbound

From (15)-(16) it is easy to check that g^1_ε, (respectively $\varepsilon g^2_\varepsilon$, $\varepsilon g^3_\varepsilon$) converges uniformly to t (resp. n, b) as ε tends to zero. We have the following strong convergences in $L^2(C, \mathbb{R})$:

$$u_{\varepsilon 1} \to U_1, \qquad \frac{u_{\varepsilon 2}}{\varepsilon} \to U_2, \qquad \frac{u_{\varepsilon 3}}{\varepsilon} \to U_3. \tag{59}$$

Let us denote $e_\varepsilon := \mathbf{e}(\mathbf{u}_\varepsilon) \circ \Phi_\varepsilon$ the strain tensor field on C. From (57), we get

$$\|e_\varepsilon\|^2_{L^2(C)} \leq 2\eta^{-1} M \varepsilon^2 \tag{60}$$

Then, $\varepsilon^{-1} e_\varepsilon$ converges weakly to some e^0. We have the following weak convergences in $L^2(C, \mathbb{R})$:

$$\begin{array}{lll} \dfrac{e_{\varepsilon 11}}{\varepsilon} \rightharpoonup E_{11}, & \dfrac{e_{\varepsilon 12}}{\varepsilon^2} \rightharpoonup E_{12}, & \dfrac{e_{\varepsilon 13}}{\varepsilon^2} \rightharpoonup E_{13}, \\ \dfrac{e_{\varepsilon 22}}{\varepsilon^3} \rightharpoonup E_{22}, & \dfrac{e_{\varepsilon 23}}{\varepsilon^3} \rightharpoonup E_{23}, & \dfrac{e_{\varepsilon 33}}{\varepsilon^3} \rightharpoonup E_{33}. \end{array} \tag{61}$$

Let us study successively the consequences of these convergences upon the asymptotic structure of the sequence u_ε.

[3]From now on we omit to precise when we extract a subsequence.

- From (19) we get

$$e_{\varepsilon 11} = \frac{\partial u_{\varepsilon 1}}{\partial x_1} - \Gamma^1_{\varepsilon 11} u_{\varepsilon 1} - \varepsilon \Gamma^2_{\varepsilon 11} \frac{u_{\varepsilon 2}}{\varepsilon} - \varepsilon \Gamma^3_{\varepsilon 11} \frac{u_{\varepsilon 3}}{\varepsilon}.$$

  From (18) one can check that $\Gamma^1_{\varepsilon 11}$, (respectively $\varepsilon\Gamma^2_{\varepsilon 11}$, $\varepsilon\Gamma^3_{\varepsilon 11}$) converges uniformly to 0 (resp. τ, ξ) as ε tends to zero. Passing to the limit in the previous equality leads to

$$\frac{\partial U_1}{\partial x_1} - \tau U_2 - \xi U_3 = 0. \tag{62}$$

- From (19) we also get

$$\frac{e_{\varepsilon 12}}{\varepsilon} = \frac{1}{2\varepsilon} \frac{\partial u_{\varepsilon 1}}{\partial x_2} + \frac{1}{2\varepsilon} \frac{\partial u_{\varepsilon 2}}{\partial x_1} - \frac{\Gamma^1_{\varepsilon 12}}{\varepsilon} u_{\varepsilon 1} - \Gamma^2_{\varepsilon 12} \frac{u_{\varepsilon 2}}{\varepsilon} - \Gamma^3_{\varepsilon 12} \frac{u_{\varepsilon 3}}{\varepsilon}.$$

  As $\varepsilon^{-1}\Gamma^1_{\varepsilon 12}$, (respectively $\Gamma^2_{\varepsilon 12}$, $\Gamma^3_{\varepsilon 12}$) converges uniformly to $-\tau$ (resp. 0, ρ) as ε tends to zero, we get by passing to the limit,

$$\frac{1}{\varepsilon} \frac{\partial u_{\varepsilon 1}}{\partial x_2} \rightharpoonup - \frac{\partial U_2}{\partial x_1} - 2\tau U_1 + 2\rho U_3. \tag{63}$$

- In the same way passing to the limit in the expression of $\varepsilon^{-1} e_{\varepsilon 12}$ leads to

$$\frac{1}{\varepsilon} \frac{\partial u_{\varepsilon 1}}{\partial x_3} \rightharpoonup - \frac{\partial U_3}{\partial x_1} + 2\xi U_1 + 2\rho U_2. \tag{64}$$

  The two last convergences show first that U_1 depends only on x_1. Then, using the Poincaré-Wirtinger inequality on each section, they show that $v_{\varepsilon 1} := \varepsilon^{-1}(u_{\varepsilon 1} - \bar{u}_{\varepsilon 1})$ is bounded in $L^2(C)$. So, $v_{\varepsilon 1}$ converges weakly to some V_1 in $L^2(C, \mathbb{R})$ and we have

$$\begin{aligned} \frac{\partial V_1}{\partial x_2} &= - \frac{\partial U_2}{\partial x_1} - 2\tau U_1 + 2\rho U_3, \\ \frac{\partial V_1}{\partial x_3} &= - \frac{\partial U_3}{\partial x_1} - 2\xi U_1 - 2\rho U_2. \end{aligned} \tag{65}$$

- For i and j in $\{2,3\}$, $\varepsilon^{-2}\, e_{\varepsilon ij}$ converges strongly to 0 while (58) implies only that

$$r_\varepsilon := \varepsilon^{-2} \left(\frac{\partial u_{\varepsilon 3}}{\partial x_2} - \frac{\partial u_{\varepsilon 2}}{\partial x_3} \right)$$

  is bounded in $L^2(C, \mathbb{R})$: it converges weakly to some r in $L^2(C, \mathbb{R})$.

  The application of Poincaré-Wirtinger inequality in each section shows that U_2 and U_3 depend only on x_1 and that the functions $v_{\varepsilon 2} := \varepsilon^{-2}(u_{\varepsilon 2} - \bar{u}_{\varepsilon 2})$ and $v_{\varepsilon 2} := \varepsilon^{-2}(u_{\varepsilon 2} - \bar{u}_{\varepsilon 2})$ are bounded in $L^2(C, \mathbb{R})$. They converge respectively to V_2 and V_3.

The application of Korn inequality in each section shows that $r_\varepsilon - \bar{r}_\varepsilon$ converges strongly to 0. So r depends only on x_1 and we have

$$V_2 = -r\,x_3, \qquad V_3 = r\,x_2. \tag{66}$$

As U_2 and U_3 depend only on x_1, equations (63)-(64) can be integrated:

$$V_1 = \left(-\frac{\partial U_2}{\partial x_1} - 2\tau U_1 + 2\rho U_3\right) x_2 + \left(-\frac{\partial U_3}{\partial x_1} - 2\xi U_1 - 2\rho U_2\right) x_3. \tag{67}$$

The asymptotic behavior of the sequence (u_ε) is described by equations (62), (66), (67) together with the fact that U_1, U_2, U_3 and r depend only on x_1.

As U_1, U_2 and U_3 depend only on x_1, it is easy to check that they coincide with the components u_1, u_2, u_3 of u in the basis (t, n, b). Indeed, g_ε^1, $\varepsilon g_\varepsilon^2$, $\varepsilon g_\varepsilon^3$ converge uniformly to t, n, b and so $u_\varepsilon = u_{\varepsilon 1} g_\varepsilon^1 + (\varepsilon^{-1}\, u_{\varepsilon 2})\varepsilon g_\varepsilon^2 + (\varepsilon^{-1}\, u_{\varepsilon 3})\varepsilon g_\varepsilon^3$ converges, like $\bar{u}_\varepsilon$, to $U_1 t + U_2 n + U_3 b$.

Now we are ready to estimate the limit energy. From (18) and (61) we get

$$\frac{\mathcal{G}_\varepsilon^{11}\, e_{\varepsilon 11}}{\varepsilon} \rightharpoonup E_{11}, \qquad \frac{\mathcal{G}_\varepsilon^{22}\, e_{\varepsilon 22}}{\varepsilon} \rightharpoonup E_{22}, \qquad \frac{\mathcal{G}_\varepsilon^{33}\, e_{\varepsilon 33}}{\varepsilon} \rightharpoonup E_{33}$$
$$\frac{\mathcal{G}_\varepsilon^{12}\, e_{\varepsilon 12}}{\varepsilon} \rightharpoonup 0, \qquad \frac{\mathcal{G}_\varepsilon^{13}\, e_{\varepsilon 13}}{\varepsilon} \rightharpoonup 0, \qquad \frac{\mathcal{G}_\varepsilon^{23}\, e_{\varepsilon 23}}{\varepsilon} \rightharpoonup 0.$$

Thus

$$\varepsilon^{-1} \mathcal{G}_\varepsilon^{ij} e_{\varepsilon ij} \rightharpoonup E_{11} + E_{22} + E_{33}. \tag{68}$$

Let us drop momentarily the summation convention. We can check in (18) that there exists a constant c_1, such that, for ε small enough and for any $i \neq j$ in $\{1, 2, 3\}$,

$$\mathcal{G}_\varepsilon^{ij} \leq c_1 \varepsilon \sqrt{\mathcal{G}_\varepsilon^{ii} \mathcal{G}_\varepsilon^{jj}}.$$

Therefore, for any matrix e, any i, j, k, l in $\{1, 2, 3\}$, if $i \neq k$ or $j \neq l$,

$$2\mathcal{G}_\varepsilon^{ik} \mathcal{G}_\varepsilon^{jl} e_{ij} e_{kl} \geq -c_1 \varepsilon \left(\mathcal{G}_\varepsilon^{ii} \mathcal{G}_\varepsilon^{jj} e_{ij} e_{ij} + \mathcal{G}_\varepsilon^{kk} \mathcal{G}_\varepsilon^{ll} e_{kk} e_{ll}\right). \tag{69}$$

Thus

$$\sum_{i,j,k,l} \mathcal{G}_\varepsilon^{ik} \mathcal{G}_\varepsilon^{jl} e_{ij} e_{kl} \geq (1 - 45 c_1 \varepsilon) \sum_{i,j} \mathcal{G}_\varepsilon^{ii} \mathcal{G}_\varepsilon^{jj} (e_{ij})^2. \tag{70}$$

Roughly speaking, one can forget in the matrix $\mathcal{G}_\varepsilon^{ij}$ the non-diagonal terms. The diagonal terms are easy to minorize:

$$\sum_{i,j,k,l} \mathcal{G}_\varepsilon^{ik} \mathcal{G}_\varepsilon^{jl} e_{ij} e_{kl} \geq (1 - c_2 \varepsilon)\Big((e_{11})^2 + \varepsilon^{-2}\,(e_{12})^2 + \varepsilon^{-2}\,(e_{13})^2$$
$$+ \varepsilon^{-4}\,(e_{22})^2 + \varepsilon^{-4}\,(e_{23})^2 + \varepsilon^{-4}\,(e_{33})^2\Big)$$

The convergences (61), (68) lead to

$$\liminf E_\varepsilon(u_\varepsilon) \geq \int_C [\mu\|E\|^2 + \frac{\lambda}{2}(tr(E))^2]\, dx \tag{71}$$

Noticing that, for any matrix E,

$$\mu\|E\|^2 + \frac{\lambda}{2}(tr(E))^2 \geq \mu\frac{2\mu+3\lambda}{2\mu+2\lambda}E_{11}^2 + 2\mu E_{12}^2 + 2\mu E_{13}^2,$$

we get

$$\liminf E_\varepsilon(u_\varepsilon) \geq \int_C [\frac{Y}{2}E_{11}^2 + 2\mu E_{12}^2 + 2\mu E_{13}^2]\, dx \tag{72}$$

We now need to study E_{11}, E_{12}, E_{13}. Let us begin by the two last ones.

- We have

$$\begin{aligned} \frac{e_{\varepsilon 12}}{\varepsilon^2} \quad &= \frac{1}{2\varepsilon^2}\frac{\partial u_{\varepsilon 1}}{\partial x_2} + \frac{1}{2\varepsilon^2}\frac{\partial u_{\varepsilon 2}}{\partial x_1} - \frac{\Gamma_{12}^1}{\varepsilon^2}u_{\varepsilon 1} - \frac{\Gamma_{12}^2}{\varepsilon^2}u_{\varepsilon 2} - \frac{\Gamma_{12}^3}{\varepsilon^2}u_{\varepsilon 3}, \\ &= \frac{1}{2}\frac{\partial v_{\varepsilon 2}}{\partial x_1} + \tau\frac{1-j_\varepsilon}{\varepsilon j_\varepsilon}\bar{u}_{\varepsilon 1} - \frac{\Gamma_{12}^1}{\varepsilon}v_{\varepsilon 1} - \frac{\Gamma_{12}^2}{\varepsilon}\frac{u_{\varepsilon 2}}{\varepsilon} \\ &\qquad - \frac{\rho\tau x_2}{j_\varepsilon}\frac{\bar{u}_{\varepsilon 3}}{\varepsilon} - \Gamma_{12}^3 v_{\varepsilon 3} + h_{\varepsilon 3}. \end{aligned}$$

  where

$$h_{\varepsilon 3} := \frac{1}{2\varepsilon^2}\frac{\partial \bar{u}_{\varepsilon 2}}{\partial x_1} + \frac{1}{2\varepsilon^2}\frac{\partial u_{\varepsilon 1}}{\partial x_2} + \frac{\tau}{\varepsilon}\bar{u}_{\varepsilon 1} - \frac{\rho}{\varepsilon^2}\bar{u}_{\varepsilon 3}. \tag{73}$$

  As every other terms in this equality converge, then $h_{\varepsilon 3}$ also converges weakly to some h_3 and, passing to the limit, we get

$$E_{12} = -\frac{1}{2}r' x_3 + \tau(\tau x_2 + \xi x_3)u_1 + \tau V_1 + \rho\tau x_3 u_2 - \rho\tau x_2 u_3 - \rho V_3 + h_3. \tag{74}$$

  Using the relations (62) (66) and (66), E_{12} takes the form $x_3 k_3 + \ell_3$ where

$$k_3 := -\tau(u_3' + \xi u_1 + \rho u_2) - \frac{1}{2}r', \tag{75}$$

$$\ell_3 := h_3 - x_2\big(\tau(u_2' + \tau u_1 - \rho u_3) + \rho r\big). \tag{76}$$

- In the same way

$$\begin{aligned} \frac{e_{\varepsilon 13}}{\varepsilon^2} &= \frac{1}{2}\frac{\partial v_{\varepsilon 3}}{\partial x_1} + \xi\frac{1-j_\varepsilon}{\varepsilon j_\varepsilon}\bar{u}_{\varepsilon 1} - \frac{\Gamma_{13}^1}{\varepsilon}v_{\varepsilon 1} \\ &\qquad - \frac{\rho\xi x_3}{j_\varepsilon}\frac{\bar{u}_{\varepsilon 2}}{\varepsilon} - \frac{\Gamma_{13}^2}{\varepsilon}v_{\varepsilon 2} - \frac{\Gamma_{13}^3}{\varepsilon}\frac{u_{\varepsilon 3}}{\varepsilon} + h_{\varepsilon 2}. \end{aligned}$$

  where

$$h_{\varepsilon 2} := \frac{1}{2\varepsilon^2}\frac{\partial \bar{u}_{\varepsilon 3}}{\partial x_1} + \frac{1}{2\varepsilon^2}\frac{\partial u_{\varepsilon 1}}{\partial x_3} + \frac{\xi}{\varepsilon}\bar{u}_{\varepsilon 1} + \frac{\rho}{\varepsilon^2}\bar{u}_{\varepsilon 2}. \tag{77}$$

As every other terms in this equality converge, then $h_{\varepsilon 2}$ also converges weakly to some h_2 and, passing to the limit, we get

$$E_{13} = \frac{1}{2} r' x_2 + \xi(\tau x_2 + \xi x_3) u_1 + \xi V_1 + \rho \xi x_3 u_2 + \rho V_2 - \xi \rho x_2 u_3 + h_2 \quad (78)$$

E_{13} takes the form $x_2 k_2 + \ell_2$ where

$$k_2 := \xi(-u_2' - \tau u_1 + \rho u_3) + \frac{1}{2} r', \quad (79)$$

$$\ell_2 := h_2 - x_3 \big(\xi(u_3' + \xi u_1 + \rho u_2) + \rho r\big). \quad (80)$$

It is important to note that k_2 and k_3 depend only on x_1 and that $h_{\varepsilon 2}$ and $h_{\varepsilon 3}$ are linked by

$$\frac{\partial h_{\varepsilon 2}}{\partial x_2} = \frac{1}{2\varepsilon^2} \frac{\partial^2 u_{\varepsilon 1}}{\partial x_2 \partial x_3} = \frac{\partial h_{\varepsilon 3}}{\partial x_3}. \quad (81)$$

So, in the sense of distributions, $\frac{\partial h_2}{\partial x_2} = \frac{\partial h_3}{\partial x_3}$, and so

$$\frac{\partial \ell_2}{\partial x_2} = \frac{\partial \ell_3}{\partial x_3} \quad (82)$$

We obtain

$$\begin{aligned} \int_C (E_{12}^2 + E_{13}^2)\, dx &= \int_0^L \Big(\int_\omega ((x_2 k_2 + \ell_2)^2 + (x_2 k_2 + \ell_2)^2)\, dx_2\, dx_3 \Big)\, dx_1 \\ &\geq \int_0^L G(k_2, k_3)\, dx_1 \end{aligned}$$

where $G(k_2, k_3)$ is the infimum, over all $(\ell_2, \ell_3) \in L^2(\omega, \mathbb{R}^2)$ satisfying (82), of

$$G(k_2, k_3) := \inf \left\{ \int_\omega ((x_2 k_2 + \ell_2)^2 + (x_2 k_2 + \ell_2)^2)\, dx_2\, dx_3 \right\}. \quad (83)$$

It is easy to check that $G(k_2, k_3) = \left(\frac{k_3 - k_2}{2}\right)^2 G(1, -1)$. As ω is simply connected, a density argument shows that $G(1, -1)$ coincides with G defined by (9). Recognizing in $k_2 - k_3$ the quantity q_1 defined by (24), we get

$$\int_C 2\mu(E_{12}^2 + 2\mu E_{13}^2)\, dx \geq \frac{1}{2} \int_0^L \mu\, G\, q_1^2\, dx_1. \quad (84)$$

- Let us now focus on E_{11}.

$$\begin{aligned} \frac{e_{\varepsilon 11}}{\varepsilon} = \frac{\partial v_{\varepsilon 1}}{\partial x_1} - \frac{\Gamma_{11}^1}{\varepsilon} u_{\varepsilon 1} - \left(\Gamma_{11}^2 - \frac{\tau}{\varepsilon}\right) \frac{u_{\varepsilon 2}}{\varepsilon} \\ - \left(\Gamma_{11}^3 - \frac{\xi}{\varepsilon}\right) \frac{u_{\varepsilon 3}}{\varepsilon} - \varepsilon \Gamma_{11}^2 v_{\varepsilon 2} - \varepsilon \Gamma_{11}^3 v_{\varepsilon 3} + h_{\varepsilon 1}, \end{aligned}$$

where

$$h_{\varepsilon 1} := \frac{1}{\varepsilon}\frac{\partial \bar{u}_{\varepsilon 1}}{\partial x_1} - \frac{\tau}{\varepsilon^2}\bar{u}_{\varepsilon 2} - \frac{\xi}{\varepsilon^2}\bar{u}_{\varepsilon 3}. \tag{85}$$

As every other terms in this equality converge then, $h_{\varepsilon 1}$ also converges to some h_1 and, passing to the limit, we have

$$\begin{aligned} E_{11} = \frac{\partial V_1}{\partial x_1} &+ \Big(\tau' x_2 + \xi' x_3 + \rho(\xi x_2 - \tau x_3)\Big) u_1 \\ &+\Big(\rho' x_3 + \rho^2 x_2 + \tau^2 x_2 + \tau\xi x_3\Big) u_2 \\ &+\Big(- \rho' x_2 + \rho^2 x_3 + \tau\xi x_2 + \xi^2 x_3\Big) u_3 \\ &-\tau V_2 - \xi V_3 + h_1 \end{aligned}$$

Then, using (62), (66) and (67), E_{11} takes the form $-x_2 q_3 + x_3 q_2 + h_1$ where q_2 and q_3, defined by (25) and (26), depend only on x_1 (recall that u_i coincides with U_i). Owing to (2), we can write:

$$\begin{aligned} \int_C E_{11}^2\, dx \quad & \geq \int_C \Big(x_2^2((q_3(x_1))^2 + x_3^2(q_2(x_1))^2 + (h_1(x_1))^2 \Big)\, dx \\ & \geq \int_0^L \Big(\int_\omega \big(x_2^2((q_3(x_1))^2 + x_3^2(q_2(x_1))^2\big)\, dx_2 dx_3 \Big)\, dx_1 \\ & \geq \int_0^L \Big(I_3(q_3(x_1))^2 + I_2(q_2(x_1))^2 \Big)\, dx_1 \end{aligned} \tag{86}$$

Collecting (84) and (86) we find that

$$\liminf E_\varepsilon(u_\varepsilon) \geq \frac{1}{2}\int_0^L \left(\mu G\, q_1^2 + Y I_2\, q_2^2 + Y I_3\, q_3^2\right)\, dx_1 \tag{87}$$

Finally, let us make some remarks:

- Equation (62) impose the condition:

$$\mathbf{u}' \cdot t = 0 \tag{88}$$

- The extension we invoked at the begining of this section, impose $u_1 = u_2 = u_3 = r = 0$ when $x_1 < 0$. So $\mathbf{u} = 0$ and $\mathbf{r} = 0$ when $x_1 < 0$.

- The fact that the right hand side of inequality (87) is bounded shows that

$$\frac{1}{2}\int_{\Omega_\varepsilon} (A \cdot (t \wedge \mathbf{u}' + \mathbf{r}t)') \cdot (t \wedge \mathbf{u}' + \mathbf{r}t)')\, d\mathcal{H}^1.$$

  is bounded. As A is uniformly cercive, $t \wedge \mathbf{u}' + \mathbf{r}t$ belongs to $H^1(\mathcal{L}, \mathrm{I\!R}^3)$. Hence $\mathbf{u}' = (t \wedge \mathbf{u}' + \mathbf{r}t) \wedge t$ belongs also to $H^1(\mathcal{L}, \mathrm{I\!R}^3)$ and $\mathbf{u}$ and $\mathbf{r}$ belong respectively to $H^2(\mathcal{L}, \rho^3)$ and $H^1(\mathcal{L}, \mathrm{I\!R})$.

- The two last remarks imply that $\mathbf{u} = \mathbf{u}' = 0$ and $\mathbf{r} = 0$ when $x_1 = 0$.

To conclude, the couple $(\mathbf{u}, \mathbf{r})$ belongs to the admissible space $(H_b^2(\mathcal{L}) \times H_b^1(\mathcal{L}))$ and

$$\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \geq \mathbf{F}(\mathbf{u}, \mathbf{r})$$

□

4.3 Upperbound

As usual in Γ-convergence proofs, we restrict our attention when proving point (iii) of Theorem 1 to a function $\mathbf{u}$ such that $\widetilde{\mathbf{F}}(\mathbf{u})$ is finite: there exists $\mathbf{r}$ such that $\widetilde{\mathbf{F}}(\mathbf{u}) = \mathbf{F}(\mathbf{u}, \mathbf{r})$. Using a density argument we restrict again our attention to regular functions $\mathbf{u}$, $\mathbf{r}$ vanishing in a neighborhood of 0.

As previously done, we associate to these functions the functions u_1, u_2, u_3, and r defined on $[0, L]$ and the quantities q_1, q_2, q_3 defined by (24)-(26). For a clearer expression of the approximating sequence, we first define on C the functions:

$$\begin{aligned}
v_1 &:= x_2(-u_2' - 2\tau u_1 + 2\rho u_3) + x_3(-u_3' - 2\xi u_1 - 2\rho u_2), \\
v_2 &:= -r x_3, \\
v_3 &:= r x_2, \\
w_1 &:= x_2^2(\rho r + \tau u_2' + \tau^2 u_1 - \rho\tau u_3) + x_3^2(\rho r + \xi u_3' + \xi^2 u_1 + \rho\xi u_2), \\
&\qquad + x_2 x_3(\xi u_2' + \tau u_3' + 2\tau\xi u_1 + \tau\rho u_2 - \xi\rho u_3) + q_1 \tilde{w}, \\
w_2 &:= \frac{\lambda}{2\lambda + 2\mu}\left(q_3 \frac{x_3^2 - x_2^2}{2} + q_2 x_2 x_3\right), \\
w_3 &:= \frac{\lambda}{2\lambda + 2\mu}\left(q_2 \frac{x_3^2 - x_2^2}{2} - q_3 x_2 x_3\right),
\end{aligned}$$

where $\tilde{w}$ is the solution of the minimisation problem (9) defining G.

Then we define $u_\varepsilon = u_{\varepsilon i} g_\varepsilon^i$ on C by setting

$$\begin{aligned}
u_{\varepsilon 1}(x_1, x_2, x_3) &:= u_1(x_1) + \varepsilon v_1(x_1, x_2, x_3) + \varepsilon^2 w_1(x_1, x_2, x_3), \\
u_{\varepsilon 2}(x_1, x_2, x_3) &:= \varepsilon u_2(x_1) + \varepsilon^2 v_2(x_1, x_2, x_3) + \varepsilon^3 w_2(x_1, x_2, x_3), \\
u_{\varepsilon 3}(x_1, x_2, x_3) &:= \varepsilon u_3(x_1) + \varepsilon^2 v_3(x_1, x_2, x_3) + \varepsilon^3 w_3(x_1, x_2, x_3).
\end{aligned}$$

It is clear that $\mathbf{u}_\varepsilon := u_\varepsilon \circ \Phi_\varepsilon^{-1}$ belongs to $H^1(\Omega_\varepsilon, \mathbb{R}^3)$. The boundary conditions are also satisfied, owing to the asumptions we made on $\mathbf{u}$, $\mathbf{r}$. Then $\mathbf{u}_\varepsilon$ is admissible. The verification of the convergence of $\bar{\mathbf{u}}_\varepsilon$ to $\mathbf{u}$ in $L^2(\mathcal{L}, \mathbb{R}^3)$ is straightforward.

Let us study successively the asymptotic behavior of each component $e_{\varepsilon ij}(u_\varepsilon)$ of the strain tensor associated to u_ε.

- A quick computation shows that

$$\frac{e_{\varepsilon 22}}{\varepsilon^3} = -\frac{\lambda}{2\lambda + 2\mu}(x_3\, q_2 - x_2\, q_3) \tag{89}$$

$$\frac{e_{\varepsilon 33}}{\varepsilon^3} = -\frac{\lambda}{2\lambda + 2\mu}(x_3\, q_2 - x_2\, q_3) \tag{90}$$

$$\frac{e_{\varepsilon 23}}{\varepsilon^3} = 0 \tag{91}$$

- Computing $e_{\varepsilon 11}$ is a bit longer. We have

$$\begin{aligned}\frac{e_{\varepsilon 11}}{\varepsilon} = \frac{1}{\varepsilon}(u_1' - \tau u_2 - \xi u_3) + \frac{\partial v_1}{\partial x_1} - \frac{\Gamma^1_{\varepsilon 11}}{\varepsilon} u_1 - \left(\Gamma^2_{\varepsilon 11} - \frac{\tau}{\varepsilon}\right) u_2 \\ - \left(\Gamma^3_{\varepsilon 11} - \frac{\xi}{\varepsilon}\right) u_3 - \tau v_2 - \xi v_3 + O(\varepsilon)\end{aligned} \tag{92}$$

As $\mathbf{F}(\mathbf{u}, \mathbf{r})$ is assumed to be finite, $\mathbf{u'} \cdot t = 0$ and so $u_1' - \tau u_2 - \xi u_3 = 0$. Terms of order ε^{-1} cancel on the right hand side of (92) and we get

$$\begin{aligned}\frac{e_{\varepsilon 11}}{\varepsilon} &= \Big(- u_2'' - 2\tau' u_1 - 2\tau(\tau u_2 + \xi u_3) + 2\rho' u_3 + 2\rho u_3'\Big)\, x_2 \\ &\quad +\Big(- u_3'' - 2\xi' u_1 - 2\xi(\tau u_2 + \xi u_3) - 2\rho' u_2 - 2\rho u_2'\Big)\, x_3 \\ &\quad +\Big(x_2 \tau' + x_3 \xi' + \rho(x_2 \xi - x_3 \tau)\Big)\, u_1 \\ &\quad +\Big((x_2 \tau + x_3 \xi)\tau + \rho' x_3 + \rho^2 x_2\Big)\, u_2 \\ &\quad +\Big((x_2 \tau + x_3 \xi)\xi - \rho' x_3 + \rho^2 x_3\Big)\, u_3 \\ &\quad +\tau r x_3 - \xi r x_2 + O(\varepsilon) \\ &= x_3\, q_2 - x_2\, q_3 + O(\varepsilon)\end{aligned} \tag{93}$$

- Computing $e_{\varepsilon 12}$ and $e_{\varepsilon 13}$ is still longer. We have

$$\begin{aligned}\frac{e_{\varepsilon 12}}{\varepsilon^2} &= \frac{1}{2\varepsilon}\left(\frac{\partial v_1}{\partial x_2} + u_2' + 2\tau u_1 - 2\rho u_3\right) + \frac{1}{2}\frac{\partial w_1}{\partial x_2} + \frac{1}{2}\frac{\partial v_2}{\partial x_1} \\ &\quad +\tau(x_2 \tau + \xi_3 \xi) u_1 + \tau \rho x_3 u_2 - \tau \rho x_2 u_3 + \tau v_1 - \rho v_3 + O(\varepsilon) \\ \frac{e_{\varepsilon 12}}{\varepsilon^2} &= \frac{q_1}{2}\left(\frac{\partial \tilde{w}}{\partial x_2} - x_3\right) + O(\varepsilon)\end{aligned} \tag{94}$$

and

$$\begin{aligned}\frac{e_{\varepsilon 13}}{\varepsilon^2} &= \frac{1}{2\varepsilon}\left(\frac{\partial v_1}{\partial x_3} + u_3' + 2\xi u_1 + 2\rho u_2\right) + \frac{1}{2}\frac{\partial w_1}{\partial x_3} + \frac{1}{2}\frac{\partial v_3}{\partial x_1} \\ &\quad +\xi(x_2 \tau + \xi_3 \xi) u_1 + \xi \rho x_3 u_2 - \xi \rho x_2 u_3 + \xi v_1 + \rho v_2 + O(\varepsilon) \\ \frac{e_{\varepsilon 13}}{\varepsilon^2} &= \frac{q_1}{2}\left(\frac{\partial \tilde{w}}{\partial x_3} + x_2\right) + O(\varepsilon)\end{aligned} \tag{95}$$

Taking into acount the order of magnitude of the $\mathcal{G}_\varepsilon^{ij}$, for any α and β in $\{1,2\}$,

$$
\begin{aligned}
e_{\varepsilon 1j}\mathcal{G}_\varepsilon^{j1} &= e_{\varepsilon 11} + O(\varepsilon^2),\\
e_{\varepsilon 1j}\mathcal{G}_\varepsilon^{j\alpha} &= \varepsilon^{-2} e_{\varepsilon 1\alpha} + O(\varepsilon),\\
e_{\varepsilon \alpha j}\mathcal{G}_\varepsilon^{j1} &= e_{\varepsilon \alpha 1} + O(\varepsilon^3),\\
e_{\varepsilon \alpha j}\mathcal{G}_\varepsilon^{j\beta} &= \varepsilon^{-2} e_{\varepsilon \alpha\beta} + O(\varepsilon^2),
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{tr(\varepsilon_\varepsilon(u_\varepsilon)}{\varepsilon} &= \frac{e_{\varepsilon 11}}{\varepsilon} + \frac{e_{\varepsilon 22}}{\varepsilon^2} + \frac{e_{\varepsilon 33}}{\varepsilon^2} + O(\varepsilon),\\
&= \frac{\mu}{\lambda+\mu}(x_3 q_2 - x_2 q_3) + O(\varepsilon), \qquad (96)\\
\frac{\|\varepsilon_\varepsilon(u_\varepsilon)\|^2}{\varepsilon^2} &= \frac{e_{\varepsilon 11}^2}{\varepsilon^2} + 2\frac{e_{\varepsilon 12}^2}{\varepsilon^4} + 2\frac{e_{\varepsilon 13}^2}{\varepsilon^4} + \frac{e_{\varepsilon 22}^2}{\varepsilon^6} + \frac{e_{\varepsilon 33}^2}{\varepsilon^6} + 2\frac{e_{\varepsilon 23}^2}{\varepsilon^6} + O(\varepsilon),\\
&= \frac{3\lambda^2 + 2\mu^2 + 4\lambda\mu}{2(\lambda+\mu)^2}(x_3 q_2 - x_2 q_3)^2\\
&\quad + \left((\frac{\partial \tilde{w}}{\partial x_3} + x_2)^2 + (\frac{\partial \tilde{w}}{\partial x_2} - x_3)^2 \right) \frac{q_1^2}{2} \; + O(\varepsilon). \qquad (97)
\end{aligned}
$$

As $\tilde{w}$ is the solution of (9), and owing to (2), we finally obtain

$$
\begin{aligned}
\lim E_\varepsilon(u_\varepsilon) &= \int_C \frac{Y}{2}(x_3 q_2 - x_2 q_3)^2 + \mu((x_2 + \frac{\partial \tilde{w}}{\partial x_3})^2 + (x_2 + \frac{\partial \tilde{w}}{\partial x_3})^2)\frac{q_1^2}{2}\, dx\\
&= \frac{1}{2}\int_0^L \left(\mu G q_1^2 + Y I_2 q_2^2 + Y I_3 q_3^2\right) dx_1 = F(u,r).
\end{aligned}
$$

$$
\lim \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{F}(\mathbf{u},\mathbf{r}) = \widetilde{\mathbf{F}}(\mathbf{u}) \qquad (98)
$$

□

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