Asymptotics of a non-planar beam in linear elasticity

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Abstract

We study the asymptotic behavior of a linear elastic material lying in
a thin tubular neighbourhood of a non planar line when the diameter of
the section tends to zero. We first estimate the Korn constant in such a
domain then we prove the convergence of the three dimensional model to
a one dimensional model. This convergence is established in the frame-
work of $\Gamma$-convergence. The resulting model is the one classically used in
mechanics. It corresponds to a non-extensional line subjected to flexion
and torsion. The torsion is an internal parameter which can eventually
by eliminated but this elimination leads to a non-local energy. Indeed the
non-planar geometry of the line couples the flexion and torsion terms.

keywords: Beam, Rod, Linear Elasticity, 3D-1D, $\Gamma$-convergence.

1 Introduction

Once this work was concluded we were advised that a similar study have been
performed by G. Griso [2]. We also notice that our way to get an estimation
for the Korn constant of the considered domain is similar to the way followed
by Mora and Muller [3] to estimate the Rigidity constant of a straight rod in
the framework of non linear elasticity. It should then be possible to extend the
results presented here to the case of non linear materials.

2 The main result

2.1 Description of the beam

First let us define the mean line of the considered beam. Let $\mathcal{L}$ be a curve
(a regular one dimensional manifold) in the physical space $\mathbb{R}^3$ and let $\varphi \in
C^3([0, \ell], \mathbb{R}^3)$ be a curvilinear parametrization of $\mathcal{L}$.
For any $x_1 \in [0, \ell]$, we denote $t(x_1) := \varphi'(x_1)$ the unit vector tangent to the curve. We complete $t(x_1)$ in order to get an orthonormal basis $(t, n, b)(x_1)$. There are many choices for such a basis: though our notations are those usually used for the Frenet basis we emphasize that the position of $(n(x_1), b(x_1))$ in the plane orthogonal to $t(x_1)$ is free. We only assume that $n$ and $b$, like $t$, belong to $C^2([0, \ell], \mathbb{R}^3)$. We can then introduce the three functions $\tau, \xi, \rho$ in $C^1([0, \ell], \mathbb{R})$ such that

$$
\begin{align*}
t'(x_1) &= \tau(x_1) n(x_1) + \xi(x_1) b(x_1) \\
n'(x_1) &= -\tau(x_1) t(x_1) + \rho(x_1) b(x_1) \\
b'(x_1) &= -\xi(x_1) t(x_1) - \rho(x_1) n(x_1)
\end{align*}
$$

(1)

Figure 1: the mean line $\mathcal{L}$ of the beam, a non planar curve.

Note that the curvature of the line can be easily recognize as $\sqrt{\tau^2 + \xi^2}$ and the torsion of the line as $\rho + \frac{\tau \xi' - \xi \tau'}{\tau^2 + \xi^2}$.

Now let us describe the section of the considered beam. Let $\omega$ be a piecewise $C^1$ domain in $\mathbb{R}^2$. It is a bounded simply-connected open set. Without loss of generality we can assume that $0$ is the inertial center of $\omega$ and that the axis $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ are the principal inertial axis of $\omega$:

$$
\int_{\omega} x_2 dx_2 dx_3 = 0, \quad \int_{\omega} x_3 dx_2 dx_3 = 0, \quad \int_{\omega} x_2 x_3 dx_2 dx_3 = 0
$$

(2)
We denote $C$ the cylinder of $\mathbb{R}^3: C := [0, \ell] \times \omega$ and $\phi_\varepsilon$ the application defined by

$$\phi_\varepsilon(x_1, x_2, x_3) := \varphi(x_1) + \varepsilon(x_2 n(x_1) + x_3 b(x_1)).$$

Throughout this paper $\varepsilon$ denotes a sequence tending to zero. For $\varepsilon$ small enough, $\phi_\varepsilon$ is a $C^2$-diffeomorphism from $C$ onto its image denoted $\Omega_\varepsilon$ (cf. figure 3). In the sequel the set $\Omega_\varepsilon$ will be referred as “the beam” and we will use in it the parametrization $\phi_\varepsilon$.

The effect of a particular choice for the basis $(n(x_1), b(x_1))$ is now clear: we can tune the way the section turns around the mean line. This is a type of “torsion” of the beam which should not be confused with the torsion of the line $L$ nor with the mechanical torsion which can result from the displacement of the beam. We emphasize the fact that, in figure 3, the beam is at rest.

2.2 Elastic energies

2.2.1 3-D linear elastic energy

Our goal is to study the behaviour of the beam $\Omega_\varepsilon$ in the framework of linear elasticity. We assume that the beam is fixed on its basis $\{x_1 = 0\}$, so any
displacement field $\mathbf{u}$ has to vanish when $x_1 = 0$. The space of admissible displacements $\mathbf{u}$ is denoted:

$$H_1^b(\Omega_\varepsilon) := \{ \mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{R}^3); \mathbf{u} = 0 \text{ when } x_1 = 0 \}.$$  

The strain tensor $\mathbf{e}(\mathbf{u})$ is the symmetric part of the gradient of $\mathbf{u}$ ($2\mathbf{e}(\mathbf{u}) := \nabla \mathbf{u} + (\nabla \mathbf{u})^t$). The elastic energy is a non-degenerated quadratic function of the strain tensor. We assume for sake of simplicity that the considered material is homogeneous and isotropic. The elastic energy $E_\varepsilon$ is then characterized by the two Lamé coefficients $\lambda$, $\mu$ with $\mu > 0$ and $3\lambda + 2\mu > 0$:

$$E_\varepsilon(\mathbf{u}) := \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} \left( \mu \|\mathbf{e}(\mathbf{u})\|^2 + \frac{\lambda}{2} (\text{tr}(\mathbf{e}(\mathbf{u})))^2 \right) d\mathbf{x}. \quad (4)$$

This energy is defined on $H_1^b(\Omega_\varepsilon)$. It is naturally extended on $L^2(\Omega_\varepsilon; \mathbb{R}^3)$ by setting $E_\varepsilon(\mathbf{u}) := +\infty$ if $\mathbf{u}$ does not belong to $H_1^b(\Omega_\varepsilon)$.

The scaling $\varepsilon^{-2}$ is needed, as we will see later, to obtain a finite energy when passing to the limit. From the mechanical point of view this scaling can be interpreted as a choice for the force unit adapted to the weak rigidity of such a thin structure.

### 2.2.2 The limit one dimensional model

The limit energy we obtain is the one classically used in mechanics. Let us describe it: it is a one dimensional model for the line $\mathcal{L}$. The displacement is described by a vector field $\mathbf{u}$ on $\mathcal{L}$ but the mechanical description is made easier by the introduction of an extra scalar field $\mathbf{r}$ on $\mathcal{L}$. In mechanics $\mathbf{r}$ is interpreted as a measure of the rotation of the sections of the beam around the mean line. The space of admissible displacements is

$$H_2^b(\mathcal{L}) := \{ \mathbf{u} \in H^2(\mathcal{L}; \mathbb{R}^3); \mathbf{u}' \cdot t = 0 \text{ along } \mathcal{L}; \mathbf{u} = \mathbf{u}' = 0 \text{ when } x_1 = 0 \}$$

while the space of admissible rotations is

$$H_1^b(\mathcal{L}) := \{ \mathbf{r} \in H^1(\mathcal{L}; \mathbb{R}); \mathbf{r} = 0 \text{ when } x_1 = 0 \}. $$

Here $\mathcal{L}$ is endowed with the one dimensional Hausdorff measure and the derivatives are relative to the curvilinear abscissa. The elastic energy $\mathbf{F}$ is then characterized by a field of positive symmetric matrices $A$:

$$\mathbf{F}(\mathbf{u}, \mathbf{r}) := \frac{1}{2} \int_{\Omega_\varepsilon} (A \cdot (t \wedge \mathbf{u}' + \mathbf{r} t)') \cdot (t \wedge \mathbf{u}' + \mathbf{r} t)') d\mathcal{H}^1. \quad (5)$$

This energy is defined on $(H_2^b(\mathcal{L}) \times H_1^b(\mathcal{L}))$ but we extend it on $L^2(\mathcal{L}; \mathbb{R}^3 \times \mathbb{R})$ by setting $\mathbf{F}(\mathbf{u}, \mathbf{r}) := +\infty$ when $(\mathbf{u}, \mathbf{r})$ does not belong to the admissible space. Note that, due to the condition $\mathbf{u}' \cdot t = 0$ in the definition of $H_2^b(\mathcal{L})$, the line is non-extensional.
In terms of the displacement only, the energy is non local: it reads
\[
\tilde{F}(u) := \min_{r \in H^1_0(\mathcal{L})} F(u, r).
\] (6)

The matrix field \( A \) is related to the geometry of the section and to the material properties by
\[
A(\varphi(x_1)) := \mu G t(x_1) \otimes t(x_1) + EI_2 n(x_1) \otimes n(x_1) + Y I_3 b(x_1) \otimes b(x_1)
\] (7)
where \( Y \) is the Young modulus of the material \( Y := \mu(3\lambda + 2\mu)(\lambda + \mu)^{-1} \) and \( I_2 \) and \( I_3 \) are the inertial moments of the section
\[
I_2 := \int_\omega (x_3)^2 dx_2 dx_3, \quad I_3 := \int_\omega (x_2)^2 dx_2 dx_3
\] (8)
and \( G \) is the infimum
\[
G := \min \left\{ \int_\omega \left( \frac{\partial \psi}{\partial x_3} + x_2 \right)^2 + \left( \frac{\partial \psi}{\partial x_2} - x_3 \right)^2 \right\}
\] (9)
which depends only on the geometry of the section. Therefore \( F \) reads
\[
F(u, r) = \frac{1}{2} \int_{\Omega_e} \left( \mu G (t \wedge u' + rt)' \cdot t \right)^2 + Y I_2 (t \wedge u' + rt)' \cdot n)^2 + \right. \\
+ \left. Y I_3 (t \wedge u' + rt)' \cdot b \right) d\mathcal{H}^1.
\] (10)

2.3 The main result

Let us first fix some notations. When no confusion can arise, we simply denote \(|D|\) the (Lebesgue or Hausdorff) measure of a set \(D\); in particular \(|\Omega_e|\), \(|\omega|\), \(|\mathcal{L}|\) denote respectively \(H^3(\Omega_e), H^2(\omega)\) and \(H^1(\mathcal{L})\). In the same way we omit to precise the measure when invoking the mean values: for instance \(\int_\omega \psi\) denotes \(|\omega|^{-1} \int_\omega \psi d\mathcal{H}^2\).

We denote \(\omega(x_1)\) the section of \(\Omega_e\) defined by \(\omega(x_1) := \Phi_e(\{x_1\} \times \omega)\) and, for any \(u \in L^2(\Omega_e)\) we denote \(\bar{u} \in L^2(\mathcal{L})\) its mean value on each section: \(\bar{u}(\varphi(x_1)) := \int_\omega \psi \).

**Theorem 1** (i) If \((u_e)\) is a sequence in \(L^2(\Omega_e, \mathbb{R}^3)\) with bounded energy (\(E_e(u_e) < M\)), then there exists a subsequence still denoted \((u_e)\) such that \(u_e\) converges in \(L^2(\mathcal{L}, \mathbb{R}^3)\).

(ii) For any sequence \((u_e)\) in \(L^2(\Omega_e, \mathbb{R}^3)\) such that \(u_e\) converges to \(u\) in \(L^2(\mathcal{L})\), we have
\[
\liminf E_e(u_e) \geq \tilde{F}(u).
\] (11)

(iii) For any \(u\) in \(L^2(\mathcal{L}, \mathbb{R}^3)\), there exists a sequence \((u_e)\) in \(L^2(\Omega_e, \mathbb{R}^3)\) such that \(u_e\) converges to \(u\) in \(L^2(\mathcal{L}, \mathbb{R}^3)\) and
\[
\limsup E_e(u_e) \leq \tilde{F}(u).
\] (12)
Remark 1 We have decided to formulate this theorem in terms of the actual displacement fields, those which arise from the physical problem and are defined on $\Omega_\varepsilon$ and $L$. One may prefer to refer to a fixed functional space. This is what is usually done in the study of straight beams \cite{4} and this is actually what we will do in the proof. Formulating the theorem in a fixed functional space has an important advantage: it can then be written in terms of $\Gamma$-convergence. A first disadvantage is that the choice of the fixed functional space is somehow arbitrary and one could then wonder whether the theorem is still valid for a different choice. A second disadvantage is the very intricate expression of the energy in the fixed space.

Remark 2 There is however a canonical way to reformulate the previous theorem in terms of $\Gamma$-convergence. Indeed let us associate to any function $u \in L^2(\Omega_\varepsilon, \mathbb{R}^3)$ the vector valued measure $|\Omega_\varepsilon|^{-1} u 1_{\Omega_\varepsilon} dx$, where $|\Omega_\varepsilon|$ and $1_{\Omega_\varepsilon}$ denote respectively the Lebesgue measure and the characteristic function of $\Omega_\varepsilon$. In the same way let us associate to any $u \in L^2(L, \mathbb{R}^3)$ the vector valued measure $|L|^{-1} u dH^1_L$, where $H^1$ denotes the one dimensional Hausdorff measure. Let us endow the space of such vector valued measures with the weak* topology. A slightly different version of the previous theorem states the relative compactness of sequences with bounded energy and the $\Gamma$-convergence of $E_\varepsilon$ to $\tilde{F}$. Indeed it is easy to check that the convergence of $u_\varepsilon$ to $u$ in the sense of measures implies, when the energy is bounded, the convergence of $\bar{u}_\varepsilon$ to $u$ in $L^2(L, \mathbb{R}^3)$.

Remark 3 Let $f$ be a continuous field of forces. A property of $\Gamma$-convergence (for details about the definition and the properties of $\Gamma$-convergence the reader can refer to \cite{1}) shows that Theorem 1 remains valid when adding in $E_\varepsilon(u_\varepsilon)$ and $\tilde{F}(u)$ respectively $-\int_{\Omega_\varepsilon} f \cdot u_\varepsilon$ and $-\int_{L} f \cdot u_1$. A second property of $\Gamma$-convergence shows that a sequence of equilibrium displacements for the beam (i.e. of minimizers of $E_\varepsilon(u_\varepsilon) - \int_{\Omega_\varepsilon} f \cdot u_\varepsilon$) converges to an equilibrium solution for the line $L$ (i.e. a minimizer of $\tilde{F}(u) - \int_{L} f \cdot u$).

2.4 Expression of energies in term of components

2.4.1 Expression of the beam energy in a fixed functional space

In order to work on a fixed functional space, we use the diffeomorphism $\Phi_\varepsilon$ as a change of variables which associates to any displacement field $u_\varepsilon \in H^1_0$, using a slightly different typography, the vector field $u_\varepsilon := u_\varepsilon \circ \Phi_\varepsilon$ defined on $C$. The space of admissible displacements $u_\varepsilon$ becomes

$$H^1_0(C) := \{ u \in H^1(C, \mathbb{R}^3); u(0, x_2, x_3) = 0, \forall (x_2, x_3) \in \omega \}. \quad (13)$$

In the same way we associate to the strain tensor $e(u_\varepsilon)$ the tensor field $e_\varepsilon$ defined on $C$ by $e_\varepsilon = e(u_\varepsilon) \circ \Phi_\varepsilon$. Note that $e_\varepsilon$ is no more the symmetric part of the gradient of $u_\varepsilon$. Let us establish the relation which links $e_\varepsilon$ and $u_\varepsilon$.

\footnote{Different choices of forces could be considered as in \cite{4} to the price of a different formulation of the theorem.}
As usual when using curvilinear coordinates, we introduce for any parameter

\[ x = (x_1, x_2, x_3) \]

the natural basis \((g_{\xi 1}(x), g_{\xi 2}(x), g_{\xi 3}(x))\) defined by

\[ g_{\xi i}(x) = \frac{\partial \phi_i}{\partial x_i}, \quad \forall i \in \{1, 2, 3\}. \]

The explicit computation of this basis leads to

\[
\begin{align*}
    g_{\xi 1}(x) &= j_2(x) t(x_1) + \varepsilon\rho(x_1)(x_2 b(x_1) - x_3 n(x_1)), \\
    g_{\xi 2}(x) &= \varepsilon n(x_1), \\
    g_{\xi 3}(x) &= b(x_1),
\end{align*}
\]

where \(\varepsilon^2 j_\xi(x)\) is the jacobian of the diffeomorphism:

\[ j_\xi(x) = 1 - \varepsilon (x_2 \tau(x_1) + x_3 \xi(x_1)). \]

This basis is not orthogonal. So it is useful to introduce the dual basis \((g^1_\xi, g^2_\xi, g^3_\xi)\)
defined by \(g^i_\xi(x) \cdot g_{\xi j}(x) = \delta^i_j\) (where \(\delta^i_j\) is the Kronecker symbol) and the metric tensor \(G^{ij}_\xi := g^i_\xi \cdot g^j_\xi\). We have:

\[
\begin{align*}
    g^1_\xi(x) &= \frac{1}{j_\xi(x)} t(x_1), \quad g^2_\xi(x) = \frac{1}{\varepsilon} n(x_1) + \frac{\rho(x_1)x_3}{j_\xi(x)} t(x_1), \\
    g^3_\xi(x) &= \frac{1}{\varepsilon} b(x_1) - \frac{\rho(x_1)x_2}{j_\xi(x)} t(x_1).
\end{align*}
\]

and

\[
\begin{align*}
    G^{11}_\xi(x) &= \frac{1}{(j_\xi(x))^2}, \quad G^{12}_\xi(x) = \frac{\rho(x_1)x_3}{(j_\xi(x))^2}, \\
    G^{22}_\xi(x) &= \frac{1}{\varepsilon^2} + \left(\frac{\rho(x_1)x_3}{j_\xi(x)}\right)^2, \quad G^{13}_\xi(x) = -\frac{\rho(x_1)x_2}{(j_\xi(x))^2}, \\
    G^{23}_\xi(x) &= -\frac{(\rho(x_1))^2 x_2 x_3}{(j_\xi(x))^2}, \quad G^{33}_\xi(x) = \frac{1}{\varepsilon^2} + \left(\frac{\rho(x_1)x_2}{j_\xi(x)}\right)^2.
\end{align*}
\]

The components of \(u_\varepsilon\) or \(e_\varepsilon\) in the dual of the natural basis are denoted respectively \(u_{\varepsilon i} := u_\varepsilon \cdot g_{\xi i}\) and \(e_{\varepsilon ij} = (e_\varepsilon \cdot g_{\xi j}) \cdot g_{\xi i}\). We have \(u_\varepsilon = u_{\varepsilon i} g^i_\varepsilon\) and \(e_\varepsilon = e_{\varepsilon ij} g^i_\varepsilon \otimes g^j_\varepsilon\).

Note that, throughout this paper, we make use of the Einstein summation convention (any repeated indices in a product have to be summed from 1 to 3).

To compute any spatial derivative using the curvilinear coordinates it is necessary to make explicit the Christoffel symbols \(\Gamma^k_{\varepsilon ij} = \Gamma^k_{\varepsilon ji} := g^k_\varepsilon \cdot \frac{\partial}{\partial x_i} (g_{ij})\):

\[
\begin{align*}
    \Gamma^{1}_{\varepsilon 11}(x) &= -\frac{\varepsilon}{j_\xi(x)} \left( x_2 \tau'(x_1) + x_3 \xi'(x_1) + \rho(x_1)(x_2 \xi(x_1) - x_3 \tau(x_1)) \right), \\
    \Gamma^{2}_{\varepsilon 11}(x) &= \frac{j_\xi(x)}{\varepsilon} \tau(x_1) - \rho'(x_1)x_3 - (\rho(x_1))^2 x_2 + \rho(x_1)x_3 \Gamma^{1}_{\varepsilon 11}(x), \\
    \Gamma^{3}_{\varepsilon 11}(x) &= \frac{j_\xi(x)}{\varepsilon} \xi(x_1) + \rho'(x_1)x_2 - (\rho(x_1))^2 x_3 - \rho(x_1)x_2 \Gamma^{1}_{\varepsilon 11}(x),
\end{align*}
\]
\[ \Gamma_{c12}^1(x) = -\frac{\epsilon \tau(x_1)}{j_c(x)}, \quad \Gamma_{c12}^2 = -\frac{\epsilon \tau(x_1)\rho(x_1) x_3}{j_c(x)}, \]
\[ \Gamma_{c13}^3(x) = \rho(x_1)(1 + \frac{\epsilon \tau(x_1) x_2}{j_c(x)}), \quad \Gamma_{c13}^1(x) = -\frac{\epsilon \xi(x_1)}{j_c(x)}, \]
\[ \Gamma_{c13}^2(x) = -\rho(x_1)(1 + \frac{\epsilon \xi(x_1) x_3}{j_c(x)}), \quad \Gamma_{c13}^3(x) = \frac{\epsilon \xi(x_1)\rho(x_1) x_2}{j_c(x)}, \]
\[ \Gamma_{c22}^i(x) = \Gamma_{c33}^i(x) = \Gamma_{c23}^i(x) = 0, \quad \forall i \in \{1, 2, 3\} \quad (18) \]

The relation between \( e_c \) and \( u_c \) is then
\[ 2e_{cij}(u_c) = \frac{\partial u_{cij}}{\partial x_j} + \frac{\partial u_{cij}}{\partial x_i} - 2\Gamma_{cij}^{\mu} u_{c\mu k} \quad (19) \]

Throughout the change of variables, the functional \( E_c \) is transformed in
\[ E_c(u) := \frac{1}{\epsilon^2} \int_{\mathcal{C}} \left( \mu \|e_c(u)\|^2 + \frac{\lambda}{2} (tr(e_c(u)))^2 \right)(x) \frac{1}{j_c(x)} dx. \quad (20) \]

To explicit the functional \( E_c \), we need to compute \( \|e_c\|^2 \) and \( tr(e_c) \) in term of the components of \( e_c \). We get
\[ \|e_c\|^2 = \|e_{cij} g_{cij} \otimes g_{cij}\|^2 = e_{cij} e_{cijk} G_{cij}^k, \]
\[ (tr(e_c))^2 = (e_{cij} tr(g_{cij} \otimes g_{cij}))^2 = e_{cij} e_{cijk} G_{cij}^k G_{cij}^k. \]

Denoting
\[ R_{cij}^{ijkl} := 2\mu G_{cij}^k G_{cij}^l + \lambda G_{cij}^k G_{cij}^l \quad (21) \]

(which is nothing else but the rigidity tensor expressed in the natural basis), we have
\[ E_c(u) = \frac{1}{2\epsilon^2} \int_{\mathcal{C}} R_{cij}^{ijkl} e_{cij} e_{cijk} \frac{1}{j_c(x)} dx. \quad (22) \]

It is not very attractive at this point to replace \( R_c \) using (21) where the \( G_{cij}^k \) are given by (18) and to replace \( e_{cij} \) using (19) in which the symbols \( \Gamma_{cij}^k \) are given by (18). It is clear that the expression of \( E_c \) is very intricate!

### 2.4.2 Expression of the limit energy in terms of components

In a natural way, we associate to any \((u, r) \in L^2(\mathcal{L}, \mathbb{R}^4)\) a field \((u, r) \in L^2([0, L], \mathbb{R}^4)\) by setting \( u = u \circ \varphi \) and \( r = r \circ \varphi \). We denote \((u_1, u_2, u_3)\) the components of \( u \) in the basis \((t, n, b)\). The explicit computation of the components of \((t \wedge u' + rt')\)' is straightforward and the functional \( F \) is transformed in
\[ F(u, r) = \frac{1}{2} \int_0^L \left( \mu G q_1^2 + Y I_2 q_2^2 + Y I_3 q_3^2 \right) dx_1 \quad (23) \]

where
\[ q_1 := r' + \tau u_3' - \xi u_2' + \rho (\tau u_2 + \xi u_3), \quad (24) \]
\[ q_2 := \tau r - u_3' - 2\rho u_2' - (\xi' + \rho \tau) u_1 - (\rho' + \xi \tau) u_2 - (\xi^2 - \rho^2) u_3, \quad (25) \]
\[ q_3 := \xi r + u_2' - 2\rho u_1' - (-\tau' + \rho \xi) u_1 - (\rho^2 - \tau^2) u_2 - (\rho' - \tau \xi) u_3, \quad (26) \]
It is also useful to note that the non-extension condition $u' \cdot t = 0$ reads

$$u'_1 - \tau u_2 - \xi u_3 = 0.$$  \hspace{1cm} (27)

### 3 Korn inequality

The first thing to study is the asymptotic behavior of the Korn constant in $\Omega_\varepsilon$. In the case of straight beams, this is usually done by comparing the expression for the energy (22) (when $\mu = 1$ and $\lambda = 0$) and the integral of the symmetric part of the gradient of $u$ and then applying Korn inequality in $C$. In the straight case $[]$, it becomes clear that, for $\varepsilon$ small enough, and for any matrix $e, R_ijkl\varepsilon e_{ij}e_{kl}$ is larger that $e_{ij}e_{ij}$. This could be also obtained (with more work) in our case.

The principal difficulty lies elsewhere: $e_{\varepsilon}(u_\varepsilon)$ is no more the symmetric part of $u_\varepsilon$. Instead $u_\varepsilon$ appears directly in the expression (19) of $e_{\varepsilon}$. Still more serious is the fact that, in this expression, some of the Christoffel symbols are very large (of order $\varepsilon^{-1}$) : a control on $e_{\varepsilon}$ does not provide any control on the symmetric part of the gradient of $u_\varepsilon$. Let us then study the asymptotic behavior of the Korn constant using a different direct method.

We establish a first lemma which states that two almost identical domains have almost the same Korn constant. For any $u \in H^1(\Omega, \mathbb{R}^N)$, we denote $e(u)$ the symmetric part of the gradient of $u$ and $\nabla a u := \nabla u - e(u)$ its skew-symmetric part. A variant of Korn inequality $[]$ states, for any domain$^{2} \Omega$ of $\mathbb{R}^N$, the existence of a constant $K_\Omega$ such that, for any $u \in H^1(\Omega, \mathbb{R}^N)$,

$$\int_\Omega \|\nabla a u - \left(\int_\Omega \nabla a u\right)\|^2 dx \leq K^2_\Omega \int_\Omega \|e(u)\|^2 dx,$$  \hspace{1cm} (28)

where $\int_\Omega \nabla a u$ denotes the mean value of $\nabla a u : \int_\Omega \nabla a u := |\Omega|^{-1} \int_\Omega \nabla a u dx$. In the following $K_\Omega$ denotes the smallest constant satisfying the previous inequality: in other words

$$K_\Omega := \sup \left\{ \|\nabla a u\|_{L^2(\Omega)}; \ u \in H^1(\Omega, \mathbb{R}^N), \int_\Omega \nabla a u dx = 0, \int_\Omega \|e(u)\|^2 dx \leq 1 \right\}$$

**Lemma 1** Let $(D_\varepsilon)$ be a sequence of domains in $\mathbb{R}^N$. Assume that there exist a domain $D$ and, for any $\varepsilon$, a diffeomorphism $\Psi_\varepsilon$ from $D_\varepsilon$ onto $D$ satisfying, at every point $x \in D_\varepsilon$, $\|\nabla \Psi_\varepsilon(x) - Id\| \leq \varepsilon$. Then there exists a constant $c > 0$, depending only on $N$ and $D$, such that for $\varepsilon$ small enough,

$$K_{D_\varepsilon} \leq K_D(1 + c \varepsilon)$$  \hspace{1cm} (29)

**Proof :** Let $u \in H^1(D_\varepsilon, \mathbb{R}^N)$ satisfying

$$\int_{D_\varepsilon} \|e(u)\|^2 dx \leq 1 \quad \text{and} \quad \int_{D_\varepsilon} \nabla a u dx = 0$$  \hspace{1cm} (30)

$^{2}$Here we call “domain” a piecewise-$C^1$, bounded and connected open set.
Let us define $v := u \circ \Psi^{-1}_\varepsilon \in H^1(D, \mathbb{R}^N)$, we have

$$u = v \circ \Psi, \quad \nabla u(x) = \nabla v(\Psi(x)) \cdot \nabla \Psi(x).$$

Hence

$$\|\nabla u(x) - \nabla v(\Psi(x))\| \leq \varepsilon \|\nabla v(\Psi(x))\|. \quad (31)$$

which gives immediately the same estimation for the symmetric and skew symmetric parts $e(u)(x) - e(v)(\Psi(x))$ and $\nabla^a u(x) - \nabla^a v(\Psi(x))$.

On the other hand, setting $c_1 := \sqrt{N} + 1$, we have, for $\varepsilon$ small enough and for any $x \in D_\varepsilon$,

$$1 - c_1 \varepsilon \leq \det(\nabla \Psi(x)) \leq 1 + c_1 \varepsilon, \quad \text{and} \quad 1 - c_1 \varepsilon \leq \det(\nabla^{-1}_\varepsilon \Psi(x)) \leq 1 + c_1 \varepsilon.$$

So for any function $\psi \in L^1(D, \mathbb{R}^m)$

$$\| \int_D \psi(y)dy - \int_{D_\varepsilon} \psi(\Psi(x))dx \| \leq c_1 \varepsilon \int_{D_\varepsilon} \|\psi(\Psi(x))\| \ dx, \quad (32)$$

or

$$\| \int_D \psi(y)dy - \int_{D_\varepsilon} \psi(\Psi(x))dx \| \leq c_1 \varepsilon \int_D \psi(y)dy, \quad (33)$$

and

$$\| \int_D \psi(y)dy - \int_{D_\varepsilon} \psi(\Psi(x))dx \| \leq 2c_1 \varepsilon \int_D \|\psi(\Psi(x))\| \ dx, \quad (34)$$

or

$$\| \int_D \psi(y)dy - \int_{D_\varepsilon} \psi(\Psi(x))dx \| \leq 2c_1 \varepsilon \int_D \psi(y)dy. \quad (35)$$

Applying inequality (34) with $\psi = \nabla^a v$, using (31), (30) and (35) we get

$$\| \int_D \nabla^a v(y)dy\| \leq \| \int_{D_\varepsilon} \nabla^a v(\Psi(x))dx\| + 2c_1 \varepsilon \int_{D_\varepsilon} \|\nabla^a v(\Psi(x))\| \ dx$$

$$\leq \| \int_{D_\varepsilon} \nabla^a u(x)dx\| + (2c_1 + 1) \varepsilon \int_{D_\varepsilon} \|\nabla v(\Psi(x))\| \ dx$$

$$\leq (2c_1 + 1) \varepsilon \int_{D_\varepsilon} \|\nabla v(\Psi(x))\| \ dx$$

$$\leq (2c_1 + 2) \varepsilon \int_D \|\nabla (y)\|dy.$$ 

Hence

$$\| \int_D \nabla^a v(y)dy\| \leq (2c_1 + 2)|D|^{-1/2} \varepsilon \|\nabla v\|_{L^2(D)}. \quad (36)$$

Applying now inequality (32) to $e(v)$, using (31), (30) and finally (33), we get

$$\int_{D} \| e(v)(y) \|^2 dy \leq (1 + c_1 \varepsilon) \int_{D} \| e(v)(\Psi_x(x)) \|^2 dx$$

$$\leq (1 + c_1 \varepsilon) \int_{D} (\| e(u)(x) \| + \varepsilon \| \nabla(v)(\Psi_x(x)) \|)^2 dx$$

$$\leq (1 + c_1 \varepsilon) \left( 1 + \varepsilon \left( \int_{D} \| \nabla(\Psi_x(x)) \|^2 dx \right) \frac{1}{2} \right)^2$$

$$\leq (1 + c_1 \varepsilon) \left( 1 + \varepsilon \left( (1 + c_1 \varepsilon) \int_{D} \| \nabla(v)(y) \|^2 dy \frac{1}{2} \right) \right)^2.$$ 

Hence

$$\| e(v) \|_{L^2(D)} \leq 1 + c_1 \varepsilon + 2 \varepsilon \| \nabla v \|_{L^2(D)}.$$ (37)

and Korn inequality in $\mathcal{D}$ leads to

$$\| \nabla^a v - \int_{D} \nabla^a v \|_{L^2(D)} \leq K_D (1 + c_1 \varepsilon + 2 \varepsilon \| \nabla v \|_{L^2(D)}).$$ (38)

Using (36) we obtain

$$\| \nabla^a v \|_{L^2(D)} \leq K_D (1 + c_1 \varepsilon) + (2 K_D + 2 c_1 + 2) \varepsilon \| \nabla v \|_{L^2(D)}.$$ (39)

and using again (37)

$$\| \nabla v \|_{L^2(D)} \leq (K_D + 1)(1 + c_1 \varepsilon) + (2 K_D + 2 c_1 + 4) \varepsilon \| \nabla v \|_{L^2(D)}.$$ (40)

Therefore $\| \nabla v \|_{L^2(D)}$ is bounded. Indeed, for $\varepsilon$ small enough,

$$\| \nabla v \|_{L^2(D)} \leq \frac{(K_D + 1)(1 + c_1 \varepsilon)}{1 - (2 K_D + 2 c_1 + 4) \varepsilon} \leq 2 K_D + 2.$$ (41)

Plugging this majoration in (39) we get

$$\| \nabla^a v \|_{L^2(D)} \leq K_D + c_2 \varepsilon,$$ (42)

where $c_2 := K_D c_1 + (2 K_D + 2 c_1 + 2)(2 K_D + 2)$. Using again (31) we have

$$\| \nabla^a u \|_{L^2(D_\varepsilon)} \leq \| \nabla^a v \circ \Psi_x \|_{L^2(D_\varepsilon)} + \varepsilon \| \nabla v \circ \Psi_x \|_{L^2(D_\varepsilon)},$$ (43)

and from (33)

$$\| \nabla^a v \circ \Psi_x \|_{L^2(D_\varepsilon)} \leq \sqrt{1 + c_1 \varepsilon} \| \nabla^a v \|_{L^2(D)}.$$ (44)

$$\| \nabla v \circ \Psi_x \|_{L^2(D_\varepsilon)} \leq \sqrt{1 + c_1 \varepsilon} \| \nabla v \|_{L^2(D)}.$$ (45)

Collecting (41), (42), (43), (44) and (45)

$$\| \nabla^a u \|_{L^2(D_\varepsilon)} \leq \sqrt{1 + c_1 \varepsilon} (K_D + 2 c_2 \varepsilon) \leq K_D (1 + c \varepsilon)$$ (46)

with $c := c_1 + 3 c_2 / K_D$. □

As the Korn inequality (28) is invariant by rescaling we easily get the following corollary:
Let us also extend any computation leads to $h$ on $\mathcal{D}_\varepsilon$ that extends $\Phi$ on $\mathcal{D}_\varepsilon$ onto $\mathcal{D}$ satisfying, at every point $x \in \mathcal{D}_\varepsilon$, $\|\nabla \Phi(x) - k_\varepsilon \text{Id}\| \leq \varepsilon$.

Now we can state a Korn theorem for $\Omega_\varepsilon$.

**Theorem 2** There exists a constant $K$ depending only on $\omega$ and $L$ such that, for $\varepsilon$ small enough and for any $u \in H^1_b(\Omega_\varepsilon)$,

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq \frac{K}{\varepsilon} \|\varepsilon(u)\|_{L^2(\Omega_\varepsilon)}$$

(47)

**Proof**: In order to take easily into account the boundary condition, let us extend (without changing the notations) the domain $\Omega_\varepsilon$ by considering a suitable extension of $\Phi_\varepsilon$ on $[-a_\varepsilon, L] \times \omega$ where $a_\varepsilon$ is chosen in $[-2\varepsilon, -\varepsilon]$ in such a way that $\varepsilon^{-1}(L + a_\varepsilon)$ is an integer (denoted $n_\varepsilon$).

Let us also extend any $u \in H^1_b(\Omega_\varepsilon)$ by setting $u = 0$ on the new part $\{x_1 \leq 0\}$.

We split the domain in $n_\varepsilon$ parts by defining, for any $i \in \{1, \ldots, n_\varepsilon\}$,

$$\Omega_i := \tilde{\Phi}_\varepsilon([a_\varepsilon + (i - 1)\varepsilon, a_\varepsilon + i\varepsilon] \times \omega)$$

(48)

Defining $h_i^\varepsilon$ by $h_i^\varepsilon(x_1, x_2, x_3) := (\varepsilon^{-1}(x_1 - a_\varepsilon) - (i - 1), x_2, x_3)$, the application $h_i^\varepsilon \circ \Phi_\varepsilon^{-1}$ is a diffeomorphism from $\Omega_i^\varepsilon$ onto the cylinder $[0, 1] \times \omega$. An explicit computation leads to $\|\nabla (\Phi_\varepsilon \circ (h_i^\varepsilon)^{-1} - \varepsilon \text{Id})\| \leq \varepsilon d \sqrt{r^2 + \xi^2 + p^2}$ where $d$ is the diameter of $\omega$. We can then apply corollary 1 and use the same Korn constant for every part $\Omega_i^\varepsilon$. Indeed, denoting $K_1 := K_{[0, 1] \times \omega}$, we have for $\varepsilon$ small enough and for any $i \in \{1, \ldots, n_\varepsilon\}$,

$$K_{\Omega_i^\varepsilon} \leq 2K_1$$

(49)

In the same way $h_i^\varepsilon \circ \Phi_\varepsilon^{-1}$ is a diffeomorphism from $\Omega_i^\varepsilon \cup \Omega_i^\varepsilon+1$ onto the cylinder $[0, 2] \times \omega$ and, denoting $K_2 := K_{[0, 2] \times \omega}$, we have for $\varepsilon$ small enough and for any $i \in \{1, \ldots, n_\varepsilon - 1\}$,

$$K_{\Omega_i^\varepsilon \cup \Omega_i^\varepsilon+1} \leq 2K_2$$

(50)

Let us introduce the mean rotation of each part $r_i^\varepsilon := \int_{\Omega_i^\varepsilon} \nabla^a u$ and the piecewise constant function $r := \sum_{i=1}^{n_\varepsilon} r_i^\varepsilon 1_{\Omega_i^\varepsilon}$. Korn inequality on each $\Omega_i^\varepsilon \cup \Omega_i^\varepsilon+1$ gives

$$\int_{\Omega_i^\varepsilon \cup \Omega_i^\varepsilon+1} \|\nabla^a u - \int_{\Omega_i^\varepsilon \cup \Omega_i^\varepsilon+1} \nabla^a u\|^2 \, dx \leq 4K_2^2 \int_{\Omega_i^\varepsilon \cup \Omega_i^\varepsilon+1} \|\varepsilon(u)\|^2 \, dx$$

Restricting the integral at the right hand side of this inequality to $\Omega_i^\varepsilon$, we get

$$\|\nabla^a u - \int_{\Omega_i^\varepsilon} \nabla^a u\|^2 \, dx \leq 4K_2^2 \int_{\Omega_i^\varepsilon} \|\varepsilon(u)\|^2 \, dx$$
which implies

\[ |\Omega^i_\varepsilon| \left\| r^i_\varepsilon - \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \nabla a u \right\|^2 \leq 4K^2 \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(u)\|^2 \, dx. \]

In the same way:

\[ |\Omega^{i+1}_\varepsilon| \left\| r^{i+1}_\varepsilon - \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \nabla a u \right\|^2 \leq 4K^2 \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(u)\|^2 \, dx \]

From which we deduce (using the fact that for any \( i \), \( |\Omega^i_\varepsilon| > \varepsilon^3 |\omega|/2 \))

\[ \left\| r^{i+1}_\varepsilon - r^i_\varepsilon \right\|^2 \leq \frac{32K^2}{|\omega|\varepsilon^3} \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(u)\|^2 \, dx \]

Using the fact that \( u = 0 \) on \( \Omega^1_\varepsilon \) and so that \( r^1_\varepsilon = 0 \), we get

\[
\left\| r^i_\varepsilon \right\|^2 \leq (i - 1) \sum_{j=1}^{i-1} \left\| r^{j+1}_\varepsilon - r^j_\varepsilon \right\|^2 \\
\leq n \varepsilon \frac{32K^2}{|\omega|\varepsilon^3} \sum_{j=1}^{i-1} \int_{\Omega^j_\varepsilon \cup \Omega^{j+1}_\varepsilon} \|e(u)\|^2 \, dx \\
\leq n \varepsilon \frac{64K^2}{|\omega|\varepsilon^3} \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(u)\|^2 \, dx \\
\leq L \frac{128K^2}{|\omega|\varepsilon^4} \int_{\Omega^i_\varepsilon \cup \Omega^{i+1}_\varepsilon} \|e(u)\|^2 \, dx
\]

Thus (using the fact that for any \( i \), \( |\Omega^i_\varepsilon| < 2\varepsilon^3 |\omega| \))

\[ \|r\|_{L^2(\Omega_\varepsilon)} \leq \frac{16K^2L}{\varepsilon} \|e(u)\|_{L^2(\Omega_\varepsilon)} \] (51)

Korn inequality on each \( \Omega^i_\varepsilon \) reads

\[
\int_{\Omega^i_\varepsilon} \left\| \nabla a u - r^i_\varepsilon \right\|^2 \, dx \leq 4K^2 \int_{\Omega^i_\varepsilon} \|e(u)\|^2 \, dx,
\]

and by summation we get

\[ \|\nabla a u - r\|_{L^2(\Omega_\varepsilon)} \leq 2K_1 \|e(u)\|_{L^2(\Omega_\varepsilon)}. \] (52)

This inequality together with (51) gives

\[ \|\nabla a u\|_{L^2(\Omega_\varepsilon)} \leq \frac{K_3}{\varepsilon} \|e(u)\|_{L^2(\Omega_\varepsilon)}, \] (53)
for any constant $K_3 > 16K_2L$, and therefore

$$\| \nabla u \|^2_{L^2(\Omega_\epsilon)} \leq \left(1 + \frac{K_3^2}{\varepsilon^2}\right) \| \varepsilon(u) \|^2_{L^2(\Omega_\epsilon)}. \quad (54)$$

Let us finally check that the Poincaré constant in $\mathcal{H}^1_b(\Omega_\epsilon)$ is bounded. Indeed, let us use the change of variables $\Phi_\varepsilon$ given by (3). The associated jacobian $\varepsilon^2 j_\varepsilon$ is given by (15). We have, for $\varepsilon$ small enough,

$$\| \nabla \Phi_\varepsilon \| \leq 2 \quad \text{and} \quad \frac{1}{2} \leq j_\varepsilon \leq 2$$

and, as $L$ is a clear upperbound for the Poincaré constant on $\mathcal{H}^1_b(C)$,

$$\| u \|^2_{L^2(\Omega_\epsilon)} \leq \int_C \| u(\Phi_\varepsilon(x)) \|^2 \varepsilon^2 j_\varepsilon \, dx \leq 2 \int_C \| u(\Phi_\varepsilon(x)) \|^2 \varepsilon^2 \, dx$$

$$\leq 2L^2 \varepsilon^2 \int_C \| \nabla(u \circ \Phi_\varepsilon)(x) \|^2 \, dx \leq 8L^2 \varepsilon^2 \int_C \| \nabla u(\Phi_\varepsilon(x)) \|^2 \, dx$$

$$\leq 8L^2 \varepsilon^2 \int_{\Omega_\varepsilon} \| \nabla u(x) \|^2 \frac{1}{\varepsilon^2 j_\varepsilon(\Phi_\varepsilon^{-1}(x))} \, dx$$

$$\leq 16L^2 \int_{\Omega_\varepsilon} \| \nabla u(x) \|^2 \, dx \quad (55)$$

Hence

$$\| u \|^2_{L^2(\Omega_\epsilon)} \leq 16L^2 \left(\frac{K_3^2}{\varepsilon^2} + 1\right) \| \varepsilon(u) \|^2_{L^2(\Omega_\epsilon)}. \quad (56)$$

The theorem is proved, choosing $K > 4LK_3$. \hfill \Box

### 4 Proof of the main theorem

#### 4.1 Compactness

First, let us extend (without changing the notations) the domain $\Omega_\epsilon$ by considering a suitable extension of $\Phi_\varepsilon$ on $[-a, L] \times \omega$ (with $a > 0$) and extend any $u \in \mathcal{H}^1_b(\Omega_\epsilon)$ by setting $u = 0$ on the new part $\{x_1 \leq 0\}$.

It is clear that it is enough to consider in the proof of points (i) or (ii) of Theorem 1 only sequences $(u_\varepsilon)$ with bounded energy $(E_\varepsilon(u_\varepsilon) \leq M)$. Moreover we can restrict our attention to a subsequence (still denote $(u_\varepsilon)$) such that $\liminf E_\varepsilon(u_\varepsilon) = \lim E_\varepsilon(u_\varepsilon)$. The statements will then be proved, when proved for some subsequence.

It is well known that, for any matrix $A$,

$$\mu \| A \|^2 + \frac{\lambda}{2} (\text{tr}(A))^2 \geq \eta \| A \|^2$$

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where \( \eta := \min\{\mu, \frac{2\mu+3\lambda}{2}\} > 0 \). Therefore the assumption \( \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leq M \) implies
\[
\|e(\mathbf{u}_\varepsilon)\|_{L^2(\Omega_\varepsilon)}^2 \leq \eta^{-1}M\varepsilon^4,
\]
and owing to Theorem 2,
\[
\|\mathbf{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 \leq K^2\eta^{-1}M\varepsilon^2,
\]
In order to work in the fixed functional space, let us use the change of variables \( \Phi_\varepsilon \) given by (3) and denote \( \mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon \circ \Phi_\varepsilon^{-1} \). Computations similar to (55), lead to
\[
\|\mathbf{u}_\varepsilon\|_{L^2(C)}^2 \leq 2\varepsilon^{-2}\|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq 2K^2\eta^{-1}M,
\]
and
\[
\|\nabla \mathbf{u}_\varepsilon\|_{L^2(C)}^2 \leq 8\varepsilon^{-2}\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq 8K^2\eta^{-1}M.
\]
The sequence \( (\mathbf{u}_\varepsilon) \) is bounded in \( H^1(C) \) and, up to a subsequence\(^3\), converges weakly to some \( \mathbf{u}^0 \) in \( H^1(C, \mathbb{R}^3) \). Then \( (\mathbf{u}_\varepsilon) \) converges strongly to \( \mathbf{u}^0 \) in \( L^2(C) \), \( (\mathbf{u}_\varepsilon) \) converges strongly to \( \mathbf{u}^0 \) in \( L^2([0,L]) \) and \( (\mathbf{u}_\varepsilon) \) converges strongly to \( \mathbf{u} := \mathbf{u}^0 \circ \varphi^{-1} \) in \( L^2(L) \). Point (i) is proved. \( \square \)

### 4.2 Lowerbound

From (15)-(16) it is easy to check that \( g^1_\varepsilon \), (respectively \( \varepsilon g^2_\varepsilon \), \( \varepsilon g^3_\varepsilon \)) converges uniformly to \( t \) (resp. \( n \), \( b \)) as \( \varepsilon \) tends to zero. We have the following strong convergences in \( L^2(C, \mathbb{R}) \):
\[
\mathbf{u}_\varepsilon^1 \rightarrow \mathbf{U}^1, \quad \frac{\mathbf{u}_\varepsilon^2}{\varepsilon} \rightarrow \mathbf{U}^2, \quad \frac{\mathbf{u}_\varepsilon^3}{\varepsilon} \rightarrow \mathbf{U}^3.
\]

Let us denote \( e_\varepsilon := e(\mathbf{u}_\varepsilon) \circ \Phi_\varepsilon \) the strain tensor field on \( C \). From (57), we get
\[
\|e_\varepsilon\|_{L^2(C)}^2 \leq 2\varepsilon^{-1}M\varepsilon^2
\]
Then, \( \varepsilon^{-1}e_\varepsilon \) converges weakly to some \( e^0 \). We have the following weak convergences in \( L^2(C, \mathbb{R}) \):
\[
\frac{e_\varepsilon^{11}}{\varepsilon} \rightharpoonup E_{11}, \quad \frac{e_\varepsilon^{12}}{\varepsilon^2} \rightharpoonup E_{12}, \quad \frac{e_\varepsilon^{13}}{\varepsilon^2} \rightharpoonup E_{13},
\]
\[
\frac{e_\varepsilon^{22}}{\varepsilon^3} \rightharpoonup E_{22}, \quad \frac{e_\varepsilon^{23}}{\varepsilon^3} \rightharpoonup E_{23}, \quad \frac{e_\varepsilon^{33}}{\varepsilon^3} \rightharpoonup E_{33}.
\]
Let us study successively the consequences of these convergences upon the asymptotic structure of the sequence \( \mathbf{u}_\varepsilon \).

\(^3\)From now on we omit to precise when we extract a subsequence.
From (19) we get
\[ e_{11} = \frac{\partial u_1}{\partial x_1} - \Gamma_{\varepsilon_{11}}^{1} u_{1} - \varepsilon \Gamma_{\varepsilon_{11}}^{2} \frac{u_{2}}{\varepsilon} - \varepsilon \Gamma_{\varepsilon_{11}}^{3} \frac{u_{3}}{\varepsilon}. \]

From (18) one can check that \( \Gamma_{\varepsilon_{11}}^{1} \) (respectively \( \varepsilon \Gamma_{\varepsilon_{11}}^{2}, \varepsilon \Gamma_{\varepsilon_{11}}^{3} \)) converges uniformly to 0 (resp. \( \tau, \xi \)) as \( \varepsilon \) tends to zero. Passing to the limit in the previous equality leads to
\[ \frac{\partial U_1}{\partial x_1} - \tau U_2 - \xi U_3 = 0. \]  

(62)

From (19) we also get
\[ e_{12} = \frac{1}{2\varepsilon} \frac{\partial u_2}{\partial x_1} + \frac{1}{2\varepsilon} \frac{\partial u_1}{\partial x_1} - \frac{\Gamma_{\varepsilon_{12}}^{1} u_{1}}{\varepsilon} - \frac{\Gamma_{\varepsilon_{12}}^{2} u_{2}}{\varepsilon} - \frac{\Gamma_{\varepsilon_{12}}^{3} u_{3}}{\varepsilon}. \]

As \( \varepsilon^{-1} \Gamma_{\varepsilon_{12}}^{1} \) (respectively \( \varepsilon^{-1} \Gamma_{\varepsilon_{12}}^{2}, \varepsilon^{-1} \Gamma_{\varepsilon_{12}}^{3} \)) converges uniformly to \( -\tau \) (resp. 0, \( \rho \)) as \( \varepsilon \) tends to zero, we get by passing to the limit,
\[ \frac{1}{\varepsilon} \frac{\partial u_1}{\partial x_2} \rightarrow - \frac{\partial U_2}{\partial x_1} - 2\tau U_1 + 2\rho U_3. \]  

(63)

In the same way passing to the limit in the expression of \( \varepsilon^{-1} e_{12} \) leads to
\[ \frac{1}{\varepsilon} \frac{\partial u_1}{\partial x_3} \rightarrow - \frac{\partial U_3}{\partial x_1} + 2\xi U_1 + 2\rho U_2. \]  

(64)

The two last convergences show first that \( U_1 \) depends only on \( x_1 \). Then, using the Poincaré-Wirtinger inequality on each section, they show that \( v_{\varepsilon_{1}} := \varepsilon^{-1}(u_{1} - \bar{u}_{1}) \) is bounded in \( L^{2}(C) \). So, \( v_{\varepsilon_{1}} \) converges weakly to some \( V_{1} \) in \( L^{2}(C, \mathbb{R}) \) and we have
\[ \frac{\partial V_{1}}{\partial x_2} = \frac{\partial U_{2}}{\partial x_1} - 2\tau U_1 + 2\rho U_3, \]
\[ \frac{\partial V_{1}}{\partial x_3} = \frac{\partial U_{3}}{\partial x_1} - 2\xi U_1 - 2\rho U_2. \]  

(65)

For \( i \) and \( j \) in \{2, 3\}, \( \varepsilon^{-2} e_{ij} \) converges strongly to 0 while (58) implies only that
\[ r_{\varepsilon} := \varepsilon^{-2} \left( \frac{\partial u_{2}}{\partial x_2} - \frac{\partial u_{2}}{\partial x_3} \right) \]

is bounded in \( L^{2}(C, \mathbb{R}) \): it converges weakly to some \( r \) in \( L^{2}(C, \mathbb{R}) \).

The application of Poincaré-Wirtinger inequality in each section shows that \( U_2 \) and \( U_3 \) depend only on \( x_1 \) and that the functions \( v_{\varepsilon_{2}} := \varepsilon^{-2}(u_{2} - \bar{u}_{2}) \) and \( v_{\varepsilon_{3}} := \varepsilon^{-2}(u_{3} - \bar{u}_{3}) \) are bounded in \( L^{2}(C, \mathbb{R}) \). They converge respectively to \( V_{2} \) and \( V_{3} \).
The application of Korn inequality in each section shows that \( r_\varepsilon - \bar{r}_\varepsilon \) converges strongly to 0. So \( r \) depends only on \( x_1 \) and we have

\[
V_2 = -r x_3, \quad V_3 = r x_2.
\] (66)

As \( U_2 \) and \( U_3 \) depend only on \( x_1 \), equations (63)-(64) can be integrated:

\[
V_1 = \left( -\frac{\partial U_2}{\partial x_1} - 2\tau U_1 + 2\rho U_3 \right) x_2 + \left( -\frac{\partial U_3}{\partial x_1} - 2\xi U_1 - 2\rho U_2 \right) x_3.
\] (67)

The asymptotic behavior of the sequence \( (u_\varepsilon) \) is described by equations (62), (66), (67) together with the fact that \( U_1, U_2, U_3 \) and \( r \) depend only on \( x_1 \).

As \( U_1, U_2 \) and \( U_3 \) depend only on \( x_1 \), it is easy to check that they coincide with the components \( u_1, u_2, u_3 \) of \( u \) in the basis \((t, n, b)\). Indeed, \( g_{11}^\varepsilon, \varepsilon g_{22}^\varepsilon, \varepsilon g_{33}^\varepsilon \) converge uniformly to \( t, n, b \) and so \( u_\varepsilon = u_{x1}g_{11}^\varepsilon + (\varepsilon^{-1} u_{x2})\varepsilon g_{22}^\varepsilon + (\varepsilon^{-1} u_{x3})\varepsilon g_{33}^\varepsilon \) converges, like \( \bar{u}_\varepsilon \), to \( U_1 t + U_2 n + U_3 b \).

Now we are ready to estimate the limit energy. From (18) and (61) we get

\[
\begin{align*}
\frac{G_{11}^\varepsilon}{\varepsilon} e_{111} & \rightarrow E_{11}, \\
\frac{G_{22}^\varepsilon}{\varepsilon} e_{222} & \rightarrow E_{22}, \\
\frac{G_{33}^\varepsilon}{\varepsilon} e_{333} & \rightarrow E_{33}, \\
\frac{G_{12}^\varepsilon}{\varepsilon} e_{121} & \rightarrow 0, \\
\frac{G_{13}^\varepsilon}{\varepsilon} e_{131} & \rightarrow 0, \\
\frac{G_{23}^\varepsilon}{\varepsilon} e_{231} & \rightarrow 0.
\end{align*}
\]

Thus

\[
\varepsilon^{-1} G_{ij}^\varepsilon e_{ij} \rightarrow E_{11} + E_{22} + E_{33}.
\] (68)

Let us drop momentarily the summation convention. We can check in (18) that there exists a constant \( c_1 \), such that, for \( \varepsilon \) small enough and for any \( i \neq j \) in \( \{1, 2, 3\} \),

\[
G_{ij}^\varepsilon \leq c_1 \varepsilon \sqrt{G_{ii}^\varepsilon G_{jj}^\varepsilon}.
\]

Therefore, for any matrix \( \varepsilon \), any \( i, j, k, l \) in \( \{1, 2, 3\} \), if \( i \neq k \) or \( j \neq l \),

\[
2G_{ij}^\varepsilon G_{k\ell}^\varepsilon e_{ij} e_{k\ell} \geq -c_1 \varepsilon \left( G_{ij}^\varepsilon G_{k\ell}^\varepsilon e_{ij} e_{k\ell} + G_{ii}^\varepsilon G_{jj}^\varepsilon e_{ij} e_{k\ell} + G_{kk}^\varepsilon G_{\ell\ell}^\varepsilon e_{ij} e_{k\ell} \right).
\] (69)

Thus

\[
\sum_{i,j,k,l} G_{ik}^\varepsilon G_{ij}^\varepsilon e_{ij} e_{kl} \geq (1 - 45c_1 \varepsilon) \sum_{i,j} G_{ij}^\varepsilon G_{ij}^\varepsilon (e_{ij})^2.
\] (70)

Roughly speaking, one can forget in the matrix \( G_{ij}^\varepsilon \) the non-diagonal terms. The diagonal terms are easy to minorize:

\[
\sum_{i,j,k,l} G_{ik}^\varepsilon G_{ij}^\varepsilon e_{ij} e_{kl} \geq (1 - c_2 \varepsilon) \left( (e_{11})^2 + \varepsilon^{-2} (e_{12})^2 + \varepsilon^{-2} (e_{13})^2 + \varepsilon^{-4} (e_{22})^2 + \varepsilon^{-4} (e_{23})^2 + \varepsilon^{-4} (e_{33})^2 \right)
\]

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The convergences (61), (68) lead to

$$\liminf E_\varepsilon(u_\varepsilon) \geq \int_C [\mu\|E\|^2 + \frac{\lambda}{2}(tr(E))^2] \, dx$$

(71)

Noticing that, for any matrix $E$,

$$\mu\|E\|^2 + \frac{\lambda}{2}(tr(E))^2 \geq \frac{2\mu + 3\lambda}{2\mu + 2\lambda}E_{11}^2 + 2\mu E_{12}^2 + 2\mu E_{13}^2,$$

we get

$$\liminf E_\varepsilon(u_\varepsilon) \geq \int_C \left[ \frac{\gamma}{2}E_{11}^2 + 2\mu E_{12}^2 + 2\mu E_{13}^2 \right] \, dx$$

(72)

We now need to study $E_{11}$, $E_{12}$, $E_{13}$. Let us begin by the two last ones.

- We have

$$\frac{e_{12}}{\varepsilon^2} = \frac{1}{2\varepsilon^2} \frac{\partial u_{12}}{\partial x_2} + \frac{1}{2\varepsilon^2} \frac{\partial u_{12}}{\partial x_1} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12},$$

$$= \frac{1}{2\varepsilon^2} \frac{\partial u_{12}}{\partial x_1} + \frac{1}{2\varepsilon^2} \frac{\partial u_{12}}{\partial x_2} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12}$$

where

$$h_{12} := \frac{1}{2\varepsilon^2} \frac{\partial u_{12}}{\partial x_1} + \frac{1}{2\varepsilon^2} \frac{\partial u_{12}}{\partial x_2} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12} - \frac{\Gamma_{12}}{\varepsilon^2} u_{12}.$$

As every other terms in this equality converge, then $h_{12}$ also converges weakly to some $h_{12}$ and, passing to the limit, we get

$$E_{12} = -\frac{1}{2} r' x_3 + \tau(x_2 + \xi x_3) u_1 + \tau V_1 + \rho r x_3 u_2 - \rho r x_2 u_3 - \rho V_3 + h_{12}. \quad (74)$$

Using the relations (62) (66) and (66), $E_{12}$ takes the form $x_3 k_3 + \ell_3$ where

$$k_3 := -\tau(u_3' + \xi u_1 + \rho u_2) - \frac{1}{2} r', \quad (75)$$

$$\ell_3 := h_3 - x_3 (\tau(u_2' + \tau u_1 - \rho u_3) + \rho r). \quad (76)$$

- In the same way

$$\frac{e_{13}}{\varepsilon^2} = \frac{1}{2\varepsilon^2} \frac{\partial u_{13}}{\partial x_1} + \frac{\Gamma_{13}}{\varepsilon^2} u_{13} - \frac{\Gamma_{13}}{\varepsilon^2} u_{13}$$

$$- \frac{\rho \xi x_3 u_{12}}{\varepsilon} - \frac{\Gamma_{13}}{\varepsilon} u_{12} - \frac{\Gamma_{13}}{\varepsilon} u_{12} + \frac{\Gamma_{13}}{\varepsilon} u_{12} + h_{12}.$$

where

$$h_{13} := \frac{1}{2\varepsilon^2} \frac{\partial u_{13}}{\partial x_1} + \frac{1}{2\varepsilon^2} \frac{\partial u_{13}}{\partial x_1} + \frac{\Gamma_{13}}{\varepsilon} u_{13} + \frac{\Gamma_{13}}{\varepsilon} u_{12}. \quad (77)$$
As every other terms in this equality converge, then \( h \varepsilon_2 \) also converges weakly to some \( h_2 \) and, passing to the limit, we get

\[
E_{13} = \frac{1}{2} r' x_2 + \xi (\tau x_2 + \xi x_3) u_1 + \xi V_1 + \rho \xi x_3 u_2 + \rho V_2 - \xi \rho x_2 u_3 + h_2 \quad (78)
\]

\( E_{13} \) takes the form \( x_2 k_2 + \ell_2 \) where

\[
k_2 := \xi (-u_2' - \tau u_1 + \rho u_3) + \frac{1}{2} r', \quad (79)
\]

\[
\ell_2 := h_2 - x_3 (\xi (u_3' + \xi u_1 + \rho u_2) + \rho r). \quad (80)
\]

It is important to note that \( k_2 \) and \( k_3 \) depend only on \( x_1 \) and that \( h \varepsilon_2 \) and \( h \varepsilon_3 \) are linked by

\[
\frac{\partial h \varepsilon_2}{\partial x_2} = \frac{1}{2 \varepsilon^2} \frac{\partial^2 u_{\varepsilon 1}}{\partial x_2 \partial x_3} = \frac{\partial h \varepsilon_3}{\partial x_3} \quad (81)
\]

So, in the sense of distributions, \( \frac{\partial k_2}{\partial x_2} = \frac{\partial k_3}{\partial x_3} \), and so

\[
\frac{\partial \ell_2}{\partial x_2} = \frac{\partial \ell_3}{\partial x_3} \quad (82)
\]

We obtain

\[
\int_C (E_{12}^2 + E_{13}^2) \, dx = \int_0^L \left( \int_\omega ((x_2 k_2 + \ell_2)^2 + (x_2 k_2 + \ell_2)^2) \, dx_2 \, dx_3 \right) \, dx_1 
\geq \int_0^L G(k_2, k_3) \, dx_1
\]

where \( G(k_2, k_3) \) is the infimum, over all \((\ell_2, \ell_3) \in L^2(\omega, \mathbb{R}^2)\) satisfying (82), of

\[
G(k_2, k_3) := \inf \left\{ \int_\omega ((x_2 k_2 + \ell_2)^2 + (x_2 k_2 + \ell_2)^2) \, dx_2 \, dx_3 \right\}. \quad (83)
\]

It is easy to check that \( G(k_2, k_3) = \left( \frac{k_3 - k_2}{2} \right)^2 G(1, -1) \). As \( \omega \) is simply connected, a density argument shows that \( G(1, -1) \) coincides with \( G \) defined by (9). Recognizing in \( k_2 - k_3 \) the quantity \( q_1 \) defined by (24), we get

\[
\int_C 2 \mu (E_{12}^2 + 2 \mu E_{13}^2) \, dx \geq \frac{1}{2} \int_0^L \mu G \, q_1^2 \, dx_1. \quad (84)
\]

- Let us now focus on \( E_{11} \).

\[
\frac{e_{111}}{\varepsilon} = \frac{\partial v_{\varepsilon 1}}{\partial x_1} - \frac{\Gamma_{11}^1}{\varepsilon} u_{\varepsilon 1} - \frac{(\Gamma_{11}^2 - \tau)}{\varepsilon} u_{\varepsilon 2} - \frac{(\Gamma_{11}^3 - \xi)}{\varepsilon} u_{\varepsilon 3} - \varepsilon \Gamma_{11}^2 v_{\varepsilon 2} - \varepsilon \Gamma_{11}^3 v_{\varepsilon 3} + h_{\varepsilon 1},
\]

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where

\[ h_{\varepsilon 1} := \frac{1}{\varepsilon} \frac{\partial \bar{u}_{\varepsilon 1}}{\partial x_1} - \frac{\tau}{\varepsilon^2} \bar{u}_{\varepsilon 2} - \frac{\xi}{\varepsilon^2} \bar{u}_{\varepsilon 3}. \]  

(85)

As every other terms in this equality converge then, \( h_{\varepsilon 1} \) also converges to some \( h_1 \) and, passing to the limit, we have

\[ E_{11} = \frac{\partial V_1}{\partial x_1} + \left( \tau' x_2 + \xi' x_3 + \rho (\xi x_2 - \tau x_3) \right) u_1 \\
+ \left( \rho' x_3 + \rho^2 x_2 + \tau^2 x_2 + \tau \xi x_3 \right) u_2 \\
+ \left( - \rho' x_2 + \rho^2 x_3 + \tau \xi x_2 + \xi^2 x_3 \right) u_3 \\
- \tau V_2 - \xi V_3 + h_1 \]

Then, using (62), (66) and (67), \( E_{11} \) takes the form

\[-x_2 q_3 + x_3 q_2 + h_1 \]

where \( q_2 \) and \( q_3 \), defined by (25) and (26), depend only on \( x_1 \) (recall that \( u_i \) coincides with \( U_i \)). Owing to (2), we can write:

\[
\int_C E_{11}^2 \, dx \geq \int_C \left( x_2^2 ((q_3(x_1))^2 + x_3^2 ((q_2(x_1))^2) + (h_1(x_1))^2) \right) \, dx \\
\geq \int_0^L \left( \int_0^\omega \left( x_2^2 ((q_3(x_1))^2 + x_3^2 ((q_2(x_1))^2) \, dx_2 dx_3 \right) \right) \, dx_1 \\
\geq \int_0^L \left( I_3(q_3(x_1))^2 + I_2(q_2(x_1))^2 \right) \, dx_1 \]  

(86)

Collecting (84) and (86) we find that

\[ \liminf E_{\varepsilon}(u_{\varepsilon}) \geq \frac{1}{2} \int_0^L \left( \mu G q_1^2 + Y I_2 q_2^2 + Y I_3 q_3^2 \right) \, dx_1 \]  

(87)

Finally, let us make some remarks:

- Equation (62) impose the condition:

\[ \mathbf{u}' \cdot t = 0 \]  

(88)

- The extension we invoked at the beginning of this section, impose \( u_1 = u_2 = u_3 = r = 0 \) when \( x_1 < 0 \). So \( u = 0 \) and \( r = 0 \) when \( x_1 < 0 \).

- The fact that the right hand side of inequality (87) is bounded shows that

\[
\frac{1}{2} \int_{\Omega_\varepsilon} \left( A \cdot (t \land \mathbf{u}' + rt)^\prime \right) \cdot (t \land \mathbf{u}' + rt)^\prime \, dH \]  

is bounded. As \( A \) is uniformly cercive, \( t \land \mathbf{u}' + rt \) belongs to \( H^1(\mathcal{L}, \mathbb{R}^3) \). Hence \( \mathbf{u}' = (t \land \mathbf{u}' + rt) \land t \) belongs also to \( H^1(\mathcal{L}, \mathbb{R}^3) \) and \( u \) and \( r \) belong respectively to \( H^2(\mathcal{L}, \rho^3) \) and \( H^1(\mathcal{L}, \mathbb{R}) \).
The two last remarks imply that \( u = u' = 0 \) and \( r = 0 \) when \( x_1 = 0 \).

To conclude, the couple \((u, r)\) belongs to the admissible space \((H^2_0(\mathcal{L}) \times H^1_0(\mathcal{L}))\) and

\[
\liminf \mathcal{E}_\epsilon(u_\epsilon) \geq \mathcal{F}(u, r)
\]

\(\Box\)

### 4.3 Upperbound

As usual in \(\Gamma\)-convergence proofs, we restrict our attention when proving point (iii) of Theorem 1 to a function \(u\) such that \(\tilde{\phi}(u)\) is finite: there exists \(r\) such that \(\tilde{\phi}(u) = \mathcal{F}(u, r)\). Using a density argument we restrict again our attention to regular functions \(u, r\) vanishing in a neighborhood of 0.

As previously done, we associate to these functions the functions \(\tilde{u}, \tilde{r}\), defined by (24)-(26). For a clearer expression of the approximating sequence, we first define on \(C\) the functions:

\[
\begin{align*}
v_1 &:= x_3(-u_2' - 2\tau u_1 + 2\rho u_3) + x_3(-u_3' - 2\xi u_1 - 2\rho u_2), \\
v_2 &:= -rx_3, \\
v_3 &:= rx_2, \\
w_1 &:= x_3^2(\rho r + \tau u_2' + \tau^2 u_1 - \rho r u_3) + x_3^2(\rho r + \xi u_3' + \xi^2 u_1 + \rho \xi u_2), \\
&\quad + x_2 x_3(\xi u_3' + \tau u_3' + 2\tau \xi u_1 + \tau \rho u_2 - \xi \rho u_3) + q_1 \tilde{w}, \\
w_2 &:= \frac{\lambda}{2\lambda + 2\mu} \left( q_3 \frac{x_3^2 - x_2^2}{2} + q_2 x_2 x_3 \right), \\
w_3 &:= \frac{\lambda}{2\lambda + 2\mu} \left( q_2 \frac{x_3^2 - x_2^2}{2} - q_3 x_2 x_3 \right),
\end{align*}
\]

where \(\tilde{w}\) is the solution of the minimisation problem (9) defining \(G\).

Then we define \(u_\epsilon = u_\epsilon g_\epsilon^0\) on \(C\) by setting

\[
\begin{align*}
u_{\epsilon 1}(x_1, x_2, x_3) &:= u_1(x_1) + \varepsilon v_1(x_1, x_2, x_3) + \varepsilon^2 w_1(x_1, x_2, x_3), \\
u_{\epsilon 2}(x_1, x_2, x_3) &:= \varepsilon v_2(x_1) + \varepsilon^2 v_2(x_1, x_2, x_3) + \varepsilon^3 w_2(x_1, x_2, x_3), \\
u_{\epsilon 3}(x_1, x_2, x_3) &:= \varepsilon v_3(x_1) + \varepsilon^2 v_3(x_1, x_2, x_3) + \varepsilon^3 w_3(x_1, x_2, x_3).
\end{align*}
\]

It is clear that \(u_\epsilon := u_\epsilon \circ \Phi_\epsilon^{-1}\) belongs to \(H^1(\Omega_\epsilon, \mathbb{R}^3)\). The boundary conditions are also satisfied, owing to the assumptions we made on \(u, r\). Then \(u_\epsilon\) is admissible. The verification of the convergence of \(u_\epsilon\) to \(u\) in \(L^2(\mathcal{L}, \mathbb{R}^3)\) is straightforward.

Let us study successively the asymptotic behavior of each component \(c_{\epsilon i j}(u_\epsilon)\) of the strain tensor associated to \(u_\epsilon\).
• A quick computation shows that

\[
\frac{e_{22}}{\varepsilon^3} = -\frac{\lambda}{2\lambda + 2\mu}(x_3 q_2 - x_2 q_3) \quad (89)
\]

\[
\frac{e_{33}}{\varepsilon^3} = -\frac{\lambda}{2\lambda + 2\mu}(x_3 q_2 - x_2 q_3) \quad (90)
\]

\[
\frac{e_{23}}{\varepsilon^3} = 0 \quad (91)
\]

• Computing \(e_{11}\) is a bit longer. We have

\[
\frac{e_{11}}{\varepsilon} = \frac{1}{\varepsilon}(u' \cdot t - \tau u_2 - \xi u_3) + \frac{\partial v_1}{\partial x_1} - \frac{e_{11}}{\varepsilon} u_1 - \left(\frac{\Gamma^1_{11} - \tau}{\varepsilon}\right) u_2
\]

\[
- \left(\Gamma^3_{11} - \frac{\xi}{\varepsilon}\right) u_3 - \tau v_2 - \xi v_3 + O(\varepsilon) \quad (92)
\]

As \(F(u, r)\) is assumed to be finite, \(u' \cdot t = 0\) and so \(u_1' - \tau u_2 - \xi u_3 = 0\). Terms of order \(\varepsilon^{-1}\) cancel on the right hand side of (92) and we get

\[
\frac{e_{11}}{\varepsilon} = x_3 q_2 - x_2 q_3 + O(\varepsilon) \quad (93)
\]

• Computing \(e_{12}\) and \(e_{13}\) is still longer. We have

\[
\frac{e_{12}}{\varepsilon^2} = \frac{1}{2\varepsilon} \left(\frac{\partial v_1}{\partial x_2} + u' + 2\tau u_1 - 2\mu u_3\right) + \frac{1}{2\varepsilon} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)
\]

\[
+ \tau(2\tau x + 2\xi) u_1 + \tau \rho x_3 u_2 - \tau \rho x_2 u_3 + \tau v_1 - \rho v_3 + O(\varepsilon)
\]

\[
\frac{e_{12}}{\varepsilon^2} = \frac{q_1}{2} \left(\frac{\partial \tilde{w}}{\partial x_2} - x_3\right) + O(\varepsilon) \quad (94)
\]

and

\[
\frac{e_{13}}{\varepsilon^2} = \frac{1}{2\varepsilon} \left(\frac{\partial v_1}{\partial x_3} + u' + 2\xi u_1 + 2\rho u_2\right) + \frac{1}{2\varepsilon} \left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)
\]

\[
+ \xi(2\tau x + 2\xi) u_1 + \xi \rho x_3 u_2 - \xi \rho x_2 u_3 + \xi v_1 + \rho v_2 + O(\varepsilon)
\]

\[
\frac{e_{13}}{\varepsilon^2} = \frac{q_1}{2} \left(\frac{\partial \tilde{w}}{\partial x_3} + x_2\right) + O(\varepsilon) \quad (95)
\]
Taking into account the order of magnitude of the $G^ij_\varepsilon$, for any $\alpha$ and $\beta$ in $\{1, 2\}$,

\[
\begin{align*}
  e_{\varepsilon 1j} G^i_\varepsilon &= e_{\varepsilon 11} + O(\varepsilon^2), \\
  e_{\varepsilon 1j} G^j_\varepsilon &= \varepsilon^{-2} e_{\varepsilon 1\alpha} + O(\varepsilon), \\
  e_{\varepsilon 0j} G^i_\varepsilon &= e_{\varepsilon 01} + O(\varepsilon^3), \\
  e_{\varepsilon 0j} G^j_\varepsilon &= \varepsilon^{-2} e_{\varepsilon 0\beta} + O(\varepsilon^2), \\
\end{align*}
\]

Hence,

\[
\begin{align*}
  \frac{\text{tr}(\varepsilon(\varepsilon u_\varepsilon))}{\varepsilon} &= \frac{e_{\varepsilon 11}}{\varepsilon} + \frac{e_{\varepsilon 22}}{\varepsilon^2} + \frac{e_{\varepsilon 33}}{\varepsilon^2} + O(\varepsilon), \\
  &= \frac{\mu}{\lambda + \mu} (x_3 q_2 - x_2 q_3) + O(\varepsilon), \\
  \frac{||\varepsilon(\varepsilon u_\varepsilon)||^2}{\varepsilon^2} &= \frac{e_{\varepsilon 11}^2}{\varepsilon^2} + \frac{2 e_{\varepsilon 12}^2}{\varepsilon^4} + \frac{2 e_{\varepsilon 13}^2}{\varepsilon^4} + \frac{e_{\varepsilon 22}^2}{\varepsilon^6} + \frac{e_{\varepsilon 33}^2}{\varepsilon^6} + \frac{2 e_{\varepsilon 23}^2}{\varepsilon^6} + O(\varepsilon), \\
  &= \frac{3\lambda^2 + 2\mu^2 + 4\lambda\mu}{2(\lambda + \mu)^2} (x_3 q_2 - x_2 q_3)^2 \\
  &\quad + \left( (\frac{\partial \tilde{w}}{\partial x_3} + x_2)^2 + (\frac{\partial \tilde{w}}{\partial x_2} - x_3)^2 \right) \frac{q_1^2}{2} + O(\varepsilon). (96)
\end{align*}
\]

As $\tilde{w}$ is the solution of (9), and owing to (2), we finally obtain

\[
\begin{align*}
  \lim E_\varepsilon(u_\varepsilon) &= \int_C \frac{Y}{2} (x_3 q_2 - x_2 q_3)^2 + \mu ((x_2 + \frac{\partial \tilde{w}}{\partial x_3})^2 + (x_2 + \frac{\partial \tilde{w}}{\partial x_2})^2) \frac{q_1^2}{2} \, dx \\
  &= \frac{1}{2} \int_0^L \left( \mu G q_1^2 + Y I_2 q_2^2 + Y I_3 q_3^2 \right) \, dx = F(u, r). \\
  \lim E_\varepsilon(u_\varepsilon) = F(u, r) = \tilde{F}(u) (98)
\end{align*}
\]

References


